For each $\alpha > 2$ there is an infinite binary word with critical exponent α

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Abstract

The critical exponent of an infinite word $\mathbf{w}$ is the supremum of all rational numbers α such that $\mathbf{w}$ contains an α-power. We resolve an open question of Krieger and Shallit by showing that for each $\alpha > 2$ there is an infinite binary word with critical exponent α.

Keywords: Combinatorics on words, repetitions, critical exponent

1 Introduction

If α is a rational number, a word w is an *α-power* if there exist words x and x' and a positive integer n, with x' a prefix of x, such that $w = x^n x'$ and $\alpha = n + |x'|/|x|$. We refer to $|x|$ as a *period* of w. A word is *α-power-free* if none of its subwords is a β-power with $\beta \geq \alpha$; otherwise, we say the word *contains an α-power.*

The *critical exponent* of an infinite word $\mathbf{w}$ is defined as

$$\sup\{\alpha \in \mathbb{Q} \mid \mathbf{w} \text{ contains an } \alpha\text{-power}\}.$$

Critical exponents of certain classes of infinite words, such as Sturmian words [8, 10] and words generated by iterated morphisms [5, 6], have received particular attention.

Krieger and Shallit [7] proved that for every real number $\alpha > 1$, there is an infinite word with critical exponent α. As α tends to 1, the number of letters required to construct

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such words tends to infinity. However, for $\alpha > 7/3$, Shur [9] gave a construction over a binary alphabet. For $\alpha > 2$, Krieger and Shallit gave a construction over a four-letter alphabet and left it as an open problem to determine if for every real number $\alpha \in (2, 7/3]$, there is an infinite binary word with critical exponent α. Currie, Rampersad, and Shallit [3] gave examples of such words for a dense subset of real numbers α in the interval $(2, 7/3]$. In this note we resolve the question completely by demonstrating that for every real number $\alpha > 2$, there is an infinite binary word with critical exponent α.

2 Properties of the Thue-Morse morphism

In this section we present some useful properties of the *Thue-Morse morphism*; *i.e.*, the morphism μ defined by $\mu(0) = 01$ and $\mu(1) = 10$. Note that $|\mu^s(0)| = |\mu^s(1)| = 2^s$ for all $s \geq 0$.

Lemma 1. *Let s be a positive integer. Let z be a subword of $\mu^{s+1}(0) = \mu^s(01)$ with $|z| \geq 2^s$. Then z does not have period 2^s.*

Proof. Write $\mu^s(0) = a_1 a_2 \ldots a_{2^n}$, $\mu^s(1) = b_1 b_2 \ldots b_{2^n}$. One checks by induction that $a_i = 1 - b_i$ for $1 \leq i \leq 2^n$, and the result follows. □

Brandenburg [1] proved the following useful theorem, which was independently rediscovered by Shur [9].

Theorem 2 (Brandenburg; Shur). *Let w be a binary word and let $\alpha > 2$ be a real number. Then w is α-power-free if and only if $\mu(w)$ is α-power-free.*

The following sharper version of one direction of this theorem (implicit in [4]) is also useful.

Theorem 3. *Suppose $\mu(w)$ contains a subword u of period p, with $|u|/p > 2$. Then w contains a subword v of length $\lceil |u|/2 \rceil$ and period $p/2$.*

We will also have call to use the *deletion operator* δ which removes the first (left-most) letter of a word. For example, $\delta(12345) = 2345$.

3 A binary word with critical exponent α

We denote by $\mathcal{L}$ the set of factors (subwords) of words of $\mu(\{0,1\}^*)$.

Lemma 4. *Let $00v \in \mathcal{L}$, and suppose that $00v$ is α-power-free for some fixed $\alpha > 2$. Let $r = \lceil \alpha \rceil$. Suppose that $0^r v = xuy$ where u contains an α-power. Then $x = \epsilon$ and $u = 0^r$.*

Proof. Suppose that u has period p. Since $00v \in \mathcal{L}$, v begins with 1. Since $00v$ is α-power-free, we can write $u = 0^s v'$, where $x = 0^{r-s}$ for some integer s, $3 \leq s \leq r$, and v' is a prefix of v. If 0^p is not a prefix of u then the prefix of u of length p contains the subword 0001. Since $\alpha > 2$, this means that 0001 is a subword of u at least twice, so that 0001 is a subword of $00v$. This is impossible, since $00v \in \mathcal{L}$.

Therefore, 0^p is a prefix of u, and u has the form 0^t for some integer $t \geq \alpha$. This implies that u has 0^r as a prefix, so that $x = \epsilon$ and $u = 0^r$. □

Lemma 5. *Let $\alpha > 2$ be given, and let $r = \lceil \alpha \rceil$. Let s, t be positive integers, such that $s \geq 3$ and there are words $x, y \in \{0,1\}^*$ such that $\mu^s(0) = x00y$ with $|x| = t$. Suppose that $2 < r - t/2^s < \alpha$ and $00v \in \mathcal{L}$ is α-power-free. Then the following statements hold.*

1. *The word $\delta^t \mu^s(0^r v)$ has a prefix which is a β-power, where $\beta = r - t/2^s$.*

2. *Suppose that $00v$ contains a β-power of period p for some β and p. Then $\delta^t \mu^s(0^r v)$ contains a β-power of period $2^s p$.*

3. *The word $\delta^t \mu^s(0^r v)$ is α-power-free.*

Proof. We start by observing that $\mu^s(0^r)$ has period 2^s. It follows that $\delta^t \mu^s(0^r)$ is a word of length $r2^s - t$ with period 2^s, and hence is a $(r2^s - t)/2^s = \beta$-power.

Now suppose u is a β-power of period p in $00v$. Then $\mu^s(u)$ is a β-power of period $2^s p$ in $\mu^s(00v)$. However, $\mu^s(0^{r-1}v)$ is a suffix of $\delta^t \mu^s(0^r v)$, since $t < 2^s = |\mu^s(0)|$. Thus $\mu^s(u)$ is a β-power of period $2^s p$ in $\delta^t \mu^s(0^r v)$.

Next, note that $\mu^s(0^{r-1}v)$ does not contain any κ-power, $\kappa \geq \alpha$. Otherwise, by Theorem 3 and induction, $0^{r-1}v$ contains a κ-power. This is impossible by Lemma 4.

Suppose then that $\delta^t \mu^s(0^r v)$ contains a κ-power $\hat{u}$ of period q, $\kappa \geq \alpha$. Using induction and Theorem 3, $0^r v$ contains a κ-power u of period $q/2^s$. By Lemma 4, the only possibility is $u = 0^r$, and $q/2^s = 1$. Thus $q = 2^s$.

Since $00v \in \mathcal{L}$, the first letter of v is a 1. Since $\hat{u}$ has period 2^s, by Lemma 1 no subword of $\mu^s(01)$ of length greater than 2^s occurs in $\hat{u}$. We conclude that either $\hat{u}$ is a subword of $\delta^t \mu^s(0^r)$, or of $\mu^s(v)$, and hence of $\mu^s(0^{r-1}v)$. As this second case has been ruled out earlier, we conclude that $|\hat{u}| \leq |\delta^t \mu^s(0^r)| = r2^s - t$. This gives a contradiction: $\hat{u}$ is a κ-power, yet $|\hat{u}|/q \leq (r2^s - t)/2^s = \beta < \alpha$. □

By construction, $\delta^t \mu^s(0^r v)$ has the form $00\hat{v}$ where $00\hat{v} \in \mathcal{L}$.

We are now ready to prove our main theorem:

Theorem 6. *Let $\alpha > 2$ be a real number. There is a word over $\{0,1\}$ with critical exponent α.*

Proof. Call a real number $\beta < \alpha$ *obtainable* if β can be written $\beta = r - t/2^s$, where r, s, t are positive integers, $s \geq 3$, and the word obtained by removing a prefix of length t from $\mu^s(0)$ begins with 00. We note that $\mu^3(0) = 01101001$ and $\mu^3(1) = 10010110$ are of length 8, and both contain 00 as a subword; for a given $s \geq 3$ it follows that r and t can be chosen so that $\beta = r - t/2^s < \alpha$ and $|\alpha - \beta| \leq 7/2^s$; by choosing large enough s, an obtainable number β can be chosen arbitrarily close to α.

Let $\{\beta_i\}$ be a sequence of obtainable numbers converging to α. For each i write $\beta_i = r_i - t_i/2^{s_i}$, where r_i, s_i, t_i are positive integers, $s_i \geq 3$, and the word obtained by removing a prefix of length t_i from $\mu^{s_i}(0)$ begins with 00. If $00w \in \mathcal{L}$, denote by $\phi_i(w)$ the word $\delta^{t_i}\mu^{s_i}(0^{r_i}w)$.

Consider the sequence of words

$$\begin{aligned} w_1 &= \phi_1(\epsilon) \\ w_2 &= \phi_1(\phi_2(\epsilon)) \\ w_3 &= \phi_1(\phi_2(\phi_3(\epsilon))) \\ &\vdots \\ w_n &= \phi_1(\phi_2(\phi_3(\cdots(\phi_n(\epsilon))\cdots))) \\ &\vdots \end{aligned}$$

By the third part of Lemma 5, if $00w \in \mathcal{L}$ is α-power-free, then so is $\phi_i(w)$. Since 00ϵ is α-power-free, each w_i is therefore α-power-free.

By the first and second parts of Lemma 5, w_n contains β_i-powers, $i = 1, 2, \ldots, n$.

Note that ϵ is a prefix of $\phi_{n+1}(\epsilon)$, so that

$$w_n = \phi_1(\phi_2(\phi_3(\cdots(\phi_n(\epsilon))\cdots)))$$

is a prefix of

$$\phi_1(\phi_2(\phi_3(\cdots(\phi_n(\phi_{n+1}(\epsilon)))\cdots))) = w_{n+1}.$$

We may therefore let $w = \lim_{n\to\infty} w_i$.

Since every prefix of w is α-power-free, w is α-power-free but contains β_i-powers for each i. The critical exponent of w is therefore α. □

The following question raised by Krieger and Shallit remains open: for $\alpha > 1$, if α-powers are avoidable on a k-letter alphabet, does there exist an infinite word over k letters with critical exponent α? In particular, for $\alpha > \mathrm{RT}(k)$, where $\mathrm{RT}(k)$ denotes the *repetition threshold* on k letters (see [2]), does there exist an infinite word over k letters with critical exponent α? We believe that the answer is "yes".

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