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Suffix conjugates for a class of morphic subshifts

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Abstract. Let A be a finite alphabet and $f\colon A^* \to A^*$ be a morphism with an iterative fixed point $f^\omega(\alpha)$, where $\alpha \in A$. Consider the subshift $(\mathcal{X}, T)$, where $\mathcal{X}$ is the shift orbit closure of $f^\omega(\alpha)$ and $T\colon \mathcal{X} \to \mathcal{X}$ is the shift map. Let S be a finite alphabet that is in bijective correspondence via a mapping c with the set of non-empty suffixes of the images $f(a)$ for $a \in A$. Let $\mathcal{S} \subset S^{\mathbb{N}}$ be the set of infinite words $\mathbf{s} = (s_n)_{n\geq 0}$ such that $\pi(\mathbf{s}) := c(s_0) f(c(s_1)) f^2(c(s_2)) \cdots \in \mathcal{X}$. We show that if f is primitive, $f^\omega(\alpha)$ is aperiodic, and $f(A)$ is a suffix code, then there exists a mapping $H\colon \mathcal{S} \to \mathcal{S}$ such that $(\mathcal{S}, H)$ is a topological dynamical system and $\pi\colon (\mathcal{S}, H) \to (\mathcal{X}, T)$ is a conjugacy; we call $(\mathcal{S}, H)$ the *suffix conjugate* of $(\mathcal{X}, T)$. In the special case where f is the Fibonacci or Thue–Morse morphism, we show that the subshift $(\mathcal{S}, T)$ is sofic, that is, the language of $\mathcal{S}$ is regular.

1. *Introduction*

Let A be a finite alphabet and $f\colon A^* \to A^*$ a morphism with an iterative fixed point $f^\omega(\alpha) = \lim_{n\to\infty} f^n(\alpha)$. Consider the shift orbit closure $\mathcal{X}$ generated by $f^\omega(\alpha)$. If $\mathbf{x} \in \mathcal{X}$, then there exist a letter $a \in A$ and an infinite word $\mathbf{y} \in \mathcal{X}$ such that $\mathbf{x} = s f(\mathbf{y})$, where s is a non-empty suffix of $f(a)$ [**5**, Lemma 6]. This formula has been observed several times in different contexts; see [**7**] and the references therein. Since $\mathbf{y} \in \mathcal{X}$, this process can be iterated to generate an expansion

$$\mathbf{x} = s_0 f(s_1) f^2(s_2) \cdots f^n(s_n) \cdots, \tag{1}$$

where each s_n is a non-empty suffix of an image of some letter in A. In general, however, not every sequence $(s_n)_{n\geq 0}$ of suffixes gives rise to an infinite word in $\mathcal{X}$ by means of this kind of expansion. Therefore, in this paper we introduce the set $\mathcal{S}$ that consists of those $(s_n)_{n\geq 0}$ whose expansion equation (1) is in $\mathcal{X}$. Our goal is then to understand the structure of $\mathcal{S}$. By endowing $\mathcal{S}$ with the usual metric on infinite words, $\mathcal{S}$ becomes a metric space. Furthermore, $\mathcal{S}$ can be associated with a mapping $G\colon \mathcal{S} \to \mathcal{S}$ (see below) giving rise to a topological dynamical system $(\mathcal{S}, G)$ that is an extension of $(\mathcal{X}, f)$; see the discussion around equation (4). However, imposing some further restrictions on f, we obtain a much stronger result. If f is a circular morphism such that $|f^n(a)| \to \infty$ for all $a \in A$ and $f(A)$ is a suffix code (Definition 1), then there exists a mapping $H\colon \mathcal{S} \to \mathcal{S}$ such that $(\mathcal{S}, H)$ and $(\mathcal{X}, T)$, where T is the usual shift operation, are conjugates (Theorem 1). We call $(\mathcal{S}, H)$ the *suffix conjugate* of $(\mathcal{X}, T)$.

In particular, both the Fibonacci morphism $\varphi\colon \mathtt{0} \mapsto \mathtt{01},\ \mathtt{1} \mapsto \mathtt{0}$ and the Thue–Morse morphism $\mu\colon \mathtt{0} \mapsto \mathtt{01},\ \mathtt{1} \mapsto \mathtt{10}$ satisfy these conditions, and so the corresponding Fibonacci subshift $(\mathcal{X}_\varphi, T)$ and the Thue–Morse subshift $(\mathcal{X}_\mu, T)$ have suffix conjugates. In this paper we characterize the language of both subshifts and show that they are regular.

In the case of bi-infinite sequences, an encoding scheme similar to ours has been considered before. Canterini and Siegel [**4**] study certain substitutive dynamical systems in the following way. Given a two-sided subshift Ω generated by a primitive substitution σ, they define an automaton (or labeled, directed graph) as follows. The vertices of this graph are letters of the alphabet, and there is an edge from a to b labeled (p, a, s) if $\sigma(b) = pas$ for some words p and s. Elements of Ω are coded by infinite paths in this graph; the set of all such paths form a subshift D of finite type. They define a lexicographic order and a successor function (i.e. a so-called 'adic transformation') on D. Canterini and Siegel then show that this adic transformation on D is a conjugate of the shift map on Ω. Holton and Zamboni [**7**] define an identical graph and show that, for a primitive substitution σ, a sequence in Ω is primitive substitutive (i.e. a morphic image of a fixed point of a primitive morphism) if and only if the sequence is encoded by an ultimately periodic path in the graph. In these encoding schemes a conjugate is obtained for all primitive substitutions, while in our encoding scheme, primitivity alone is not sufficient; see Example 2.

See also the work by Shallit [**15**], who constructed a finite automaton that provides an encoding for the set of infinite overlap-free words.

2. *Preliminaries and generalities*

In this paper we will follow the standard notation and terminology of combinatorics on words [**1**, **11**] and symbolic dynamics [**9**, **10**].

Let A be a finite alphabet and $f : A^* \to A^*$ a morphism with an iterative fixed point $f^\omega(\alpha) = \lim_{n\to\infty} f^n(\alpha)$, where $\alpha \in A$. Let $\mathcal{X}$ be the shift orbit closure generated by $f^\omega(\alpha)$. Let S' be the set of non-empty suffixes of images of letters under f. Write $S = \{0, 1, \ldots, |S'| - 1\}$ and let $c\colon S \to S'$ be a bijection. We consider S as a finite alphabet.

If $s = s_0 s_1 \cdots s_n$ with $s_i \in S$, then we denote by $\pi(s)$ the word

$$\pi(s) = c(s_0) f(c(s_1)) f^2(c(s_2)) \cdots f^n(c(s_n)) \in A^*.$$

Then π extends to a mapping $\pi : S^{\mathbb{N}} \to A^{\mathbb{N}}$ in a natural way, and so we may define

$$\mathcal{S} = \{\mathbf{s} \in S^{\mathbb{N}} \mid \pi(\mathbf{s}) \in \mathcal{X}\}.$$

Our goal in this section is to find sufficient conditions on f so that $\mathcal{S}$ can be endowed with dynamics that yields a conjugate to $(\mathcal{X}, T)$ via the mapping π. Examples 1 and 2 below show that this task is not trivial. Such sufficient conditions are laid out in Definition 1.

If $\mathbf{x} \in \mathcal{X}$ and $\mathbf{s} \in \mathcal{S}$ such that $\pi(\mathbf{s}) = \mathbf{x}$, we say that $\mathbf{x}$ is an *expansion* of $\mathbf{s}$.

LEMMA 1. (Currie *et al* [**5**]) *For every $\mathbf{x} \in \mathcal{X}$, there exist $a \in A$, a non-empty suffix s of $f(a)$, and an infinite word $\mathbf{y} \in \mathcal{X}$ such that $\mathbf{x} = sf(\mathbf{y})$ and $a\mathbf{y} \in \mathcal{X}$. Therefore the mapping $\pi : \mathcal{S} \to \mathcal{X}$ is surjective.*

Both $A^{\mathbb{N}}$ and $S^{\mathbb{N}}$ are endowed with the usual metric

$$d((x_n)_{n\geq 0}, (y_n)_{n\geq 0}) = \frac{1}{2^n} \quad \text{where } n = \inf\{n \mid x_n \neq y_n\},$$

The following lemma is obvious.

LEMMA 2. *The mapping $\pi : \mathcal{S} \to \mathcal{X}$ is continuous.*

We denote the usual shift operation $(x_n)_{n\geq 0} \mapsto (x_{n+1})_{n\geq 0}$ in both spaces $A^{\mathbb{N}}$ and $S^{\mathbb{N}}$ by T. We have $T(\mathcal{X}) \subset \mathcal{X}$ and $f(\mathcal{X}) \subset \mathcal{X}$ by the construction of $\mathcal{X}$, and both T and f are clearly continuous on $\mathcal{X}$, so we have the topological dynamical systems $(\mathcal{X}, T)$ and $(\mathcal{X}, f)$. Note, however, that in general $T(\mathcal{S})$ is not necessarily a subset of $\mathcal{S}$, as the following contrived example shows.

Example 1. Let $f \colon A^* \to A^*$ with $A = \{\alpha, \beta, a, b\}$ be the morphism $\alpha \mapsto \alpha ab$, $a \mapsto a$, $b \mapsto ab$, and $\beta \mapsto \alpha$. Then $S' = \{\alpha ab, ab, b, a, \alpha\}$. Define $c \colon S \to S'$ by $c(0) = \alpha ab$, $c(1) = ab$, $c(2) = b$, $c(3) = a$, $c(4) = \alpha$. Then $\pi(42^\omega) = f^\omega(\alpha)$, so that $42^\omega \in \mathcal{S}$. However,

$$\pi(2^\omega) = bf(b)f^2(b)\cdots = bab\cdots .$$

Since bab does not occur in $f^\omega(\alpha)$, this shows that $T(42^\omega) \notin \mathcal{S}$, and so $\mathcal{S}$ is not closed under T.

If f is the morphism $0 \mapsto 01$, $1 \mapsto 0$, then f is called the *Fibonacci morphism* and we write $f = \varphi$. The unique fixed point of φ is denoted by $\mathbf{f}$ and it is called the *Fibonacci word*. The shift orbit closure it generates is denoted by $\mathcal{X}_\varphi$ and the pair $(\mathcal{X}_\varphi, T)$ is called the *Fibonacci subshift*.

Similarly, if f is $0 \mapsto 01$, $1 \mapsto 10$, then f is the *Thue–Morse morphism* and we write $f = \mu$. The fixed point $\mu^\omega(0)$ of μ is denoted by $\mathbf{t}$ and it is called the *Thue–Morse word*. The shift orbit closure generated by $\mathbf{t}$ is denoted by $\mathcal{X}_\mu$, and the pair $(\mathcal{X}_\mu, T)$ is called the *Thue–Morse subshift*.

Example 2. Let f be the morphism $0 \mapsto 010$, $1 \mapsto 10$. Then, using the observation that $010f(a) = \varphi^2(a)010$ for $a \in \{0, 1\}$, it is not difficult to verify that $f^\omega(0) = 01\mathbf{f}$, where $\mathbf{f}$ is the Fibonacci word. Thus the shift orbit closures of $f^\omega(0)$ and $\mathbf{f}$ coincide. The set of suffixes of $f(0)$ and $f(1)$ is $S' = \{0, 10, 010\}$, and we define a bijection $c \colon \{0, 1, 2\} \to S'$ by $c(0) = 0$, $c(1) = 10$, and $c(2) = 010$. Then it can be shown that $\pi(01^\omega) = \pi(201^\omega) = \mathbf{f}$. This shows that it is possible for two distinct words in $\mathcal{S}$ to have the same expansions, and therefore π is not always injective.

The following lemma is a straightforward consequence of the definition of π.

LEMMA 3. *Let $\mathbf{s} = s_0 s_1 s_2 \cdots$, where $s_i \in S$. Then*

$$f(\pi \circ T(\mathbf{s})) = T^{|c(s_0)|}\pi(\mathbf{s})$$

and

$$\pi(\mathbf{s}) = \pi(s_0 s_1 \cdots s_{n-1}) f^n(\pi(T^n \mathbf{s})). \tag{2}$$

For finite words $x, y \in S^*$, the above reads $\pi(xy) = \pi(x) f^{|x|}(\pi(y))$.

Note that if $s \in S$ such that $c(s) \in S'$ is a letter, then $f(c(s)) \in S'$. As this connection will be frequently referred to, we define a morphism

$$\lambda\colon S_1^* \to S^* \quad \text{with } \lambda(s) = c^{-1}(f(c(s))), \tag{3}$$

where $S_1 \subset S$ consists of those $s \in S$ for which $|c(s)| = 1$. Then, in particular, $c(\lambda(s)) = f(c(s))$.

LEMMA 4. *Let $\mathbf{s} = s_0 s_1 \cdots \in \mathcal{S}$ with $s_i \in S$, and write $\mathbf{x} = \pi(\mathbf{s}) \in \mathcal{X}$. Let $r \geq 0$ be the smallest integer, if it exists, such that $|c(s_r)| \geq 2$ and write $c(s_r) = au$, where $a \in A$ and $u \in A^+$. Then $f(\mathbf{x}) = \pi(\mathbf{t})$, where $\mathbf{t} = t_0 t_1 \cdots \in \mathcal{S}$ satisfies*

- *$t_i = \lambda(s_i)$ for $i = 0, 1, \ldots, r-1$,*
- *$t_r = c^{-1}(f(a))$,*
- *$t_{r+1} = c^{-1}(u)$, and*
- *$t_i = s_{i-1}$ for $i \geq r+2$.*

If each of $c(s_i)$ is a letter, then $f(\mathbf{x}) = \pi(\mathbf{t})$, where

$$\mathbf{t} = \lambda(s_0)\lambda(s_1) \cdots \lambda(s_n) \cdots .$$

Proof. Suppose that r exists. The identity $\mathbf{x} = \pi(\mathbf{s})$ says that

$$\mathbf{x} = c(s_0) f(c(s_1)) \cdots f^{r-1}(c(s_{r-1})) f^r(c(s_r)) f^{r+1}(c(s_{r+1})) \cdots .$$

Therefore, by writing $f(c(s_i)) = \hat{s}_i \in S'$ for $i = 0, 1, \ldots, r-1$, we see that

$$\begin{aligned} f(\mathbf{x}) &= f(c(s_0)) f^2(c(s_1)) \cdots f^r(c(s_{r-1})) f^{r+1}(c(s_r)) f^{r+2}(c(s_{r+1})) \cdots \\ &= \hat{s}_0 f(\hat{s}_1) \cdots f^{r-1}(\hat{s}_{r-1}) f^{r+1}(au) f^{r+2}(c(s_{r+1})) \cdots \\ &= \hat{s}_0 f(\hat{s}_1) \cdots f^{r-1}(\hat{s}_{r-1}) f^r(f(a)) f^{r+1}(u) f^{r+2}(c(s_{r+1})) \cdots \\ &= c(t_0) f(c(t_1)) f^2(c(t_2)) \cdots, \end{aligned}$$

where the t_i are as in the statement of the lemma. The case where r does not exist is a special case of the above. □

Let $\mathbf{s} \in \mathcal{S}$ and $\mathbf{t} \in \mathcal{S}$ be defined as in the previous lemma. This defines a mapping $G\colon \mathcal{S} \to \mathcal{S}$ for which $G(\mathbf{s}) = \mathbf{t}$, which is obviously continuous. Thus we have a topological dynamical system $(\mathcal{S}, G)$. Furthermore, by the definition of G, we have

$$f \circ \pi = \pi \circ G. \tag{4}$$

Therefore $\pi\colon (\mathcal{S}, G) \to (\mathcal{X}, f)$ is a factor map because π is surjective by Lemma 1 and continuous by Lemma 2. We can get a more concise definition for G if we extend the domain of λ defined in equation (3) to S as follows. If $s \in S \setminus S_1$, then $f(c(s)) = au$ with $a \in A$ and $u \in A^+$, and we define

$$\lambda(s) = c^{-1}(f(a)) c^{-1}(u). \tag{5}$$

Then we have, for all $\mathbf{s} \in \mathcal{S}$,

$$G(\mathbf{s}) = \begin{cases} \lambda(ps)\mathbf{t} & \text{if } \mathbf{s} = ps\mathbf{t} \text{ with } p \in S_1^* \text{ and } s \in S \setminus S_1, \\ \lambda(\mathbf{s}) & \text{if } \mathbf{s} \in S_1^{\mathbb{N}}. \end{cases} \tag{6}$$

We got this far without imposing any restrictions on f, but now we have to introduce some further concepts.

If $\mathcal{Y}$ is the shift orbit closure of some infinite word $\mathbf{x}$, then the set of finite factors of $\mathbf{x}$ is called the *language* of $\mathcal{Y}$ or $\mathbf{x}$ and denoted by $\mathcal{L}(\mathcal{Y})$ or by $\mathcal{L}(\mathbf{x})$.

If x is a finite word and y a finite or infinite word and x is a factor of y, we will express this by writing $x \subset y$. This handy notation has been used before at least in [**6**].

A key property we would like our morphism f to have is called *circularity* [**3**, **8**, **12**]. The morphism f whose fixed point generates the shift orbit closure $\mathcal{X}$ is called *circular on* $\mathcal{L}(\mathcal{X})$ if (1) f is injective on $\mathcal{L}(\mathcal{X})$ and (2) there exists a *synchronization delay* $\ell \geq 1$ such that if $w \in \mathcal{L}(\mathcal{X})$ and $|w| \geq \ell$, then it has a *synchronizing point* (w_1, w_2) satisfying the following two conditions. First, $w = w_1 w_2$. Second,

$$\text{for all } v_1, v_2 \in A^* \; [v_1 w v_2 \in f(\mathcal{L}(\mathcal{X})) \Longrightarrow v_1 w_1 \in f(\mathcal{L}(\mathcal{X})) \text{ and } w_2 v_2 \in f(\mathcal{L}(\mathcal{X}))].$$

Definition 1. We write $f \in \mathcal{N}$ to indicate that $f\colon A^* \to A^*$ with an iterative fixed point $f^{\omega}(\alpha)$ has the following properties:
(i) f is circular on the language of $f^{\omega}(\alpha)$;
(ii) the set $f(A)$ is a suffix code, i.e. no image of a letter is a suffix of another;
(iii) each letter $a \in A$ is growing, i.e. $|f^n(a)| \to \infty$ as $n \to \infty$.

In what follows, we will show that $\mathcal{N}$ contains a large and interesting class of morphisms; see Lemma 7. For that purpose we will need the concept of bilateral recognizability and Mossé's theorem† given in Lemma 5.

Let $f\colon A^* \to A^*$ be a morphism with a fixed point $\mathbf{x} = x_0 x_1 x_2 \cdots$, $x_i \in A$, and define

$$E_1 = \{0\} \cup \{|f(x_0 x_1 \cdots x_p)| : \; p \geq 0\}.$$

The morphism f is called *bilaterally recognizable* if there exists an integer $L > 0$ such that if

$$x_{i-L} x_{i-L+1} \cdots x_{i+L} = x_{j-L} x_{j-L+1} \cdots x_{j+L} \tag{7}$$

and $i \in E_1$, then $j \in E_1$.

LEMMA 5. (Mossé [**14**]) *If f is primitive and* $\mathbf{x}$ *is aperiodic, then f is bilaterally recognizable.*

The following lemma is a quite straightforward consequence of Mossé's theorem. It has been mentioned before [**3**, **8**], but without a proof, so we provide one here for completeness.

LEMMA 6. *If f is primitive,* $\mathbf{x}$ *is aperiodic, and f is injective on* $\mathcal{L}(\mathbf{x})$*, then f is circular.*

† Mossé's paper [**14**] is in French. See [**9**] for a discussion in English.

Proof. Since the injectivity of f is assumed, only condition (2) in the definition of circularity has to be proved. Let L be the constant from the bilateral recognizability given by Lemma 5, and define $\ell = 2L + \max_{a \in A} |f(a)|$. We will show that ℓ can be taken to be a synchronization delay of f. Indeed, take $w \in \mathcal{L}(\mathcal{X})$ with $|w| \geq \ell$. Then there exist u_1, u_2 such that $u_1 w u_2 \in f(\mathcal{L}(\mathcal{X}))$. Let i' be an integer such that $u_1 w u_2$ occurs in $\mathbf{x}$ at index $i' - |u_1|$; thus w occurs at index i'. The choice of ℓ implies that there exists an integer k with $L < k < |w| - L$ such that $i' + k \in E_1$. Write $i = i' + k$, $w_1 = x_{i'} \cdots x_{i-1}$, and $w_2 = x_i \cdots x_{i'+|w|-1}$. Then $w = w_1 w_2$ and $u_1 w_1, w_2 u_2 \in f(\mathcal{L}(\mathcal{X}))$ with $|w_1|, |w_2| \geq L$.

We claim that (w_1, w_2) is a synchronizing point for w. To see this, suppose that $v_1 w v_2 \in f(\mathcal{L}(\mathcal{X}))$ and let j' be an integer such that $j' - |v_1|$ is an occurrence of $v_1 w v_2$ in $\mathbf{x}$. Write $j = j' + k$. Then (7) holds and since $i \in E_1$, bilateral recognizability of f implies that $j \in E_1$; that is, $v_1 w_1 \in f(\mathcal{L}(\mathcal{X}))$ and $w_2 v_2 \in f(\mathcal{L}(\mathcal{X}))$. Therefore f is circular. □

LEMMA 7. *If f is primitive, $f(A)$ a suffix code, and $\mathbf{x}$ aperiodic, then $f \in \mathcal{N}$.*

Proof. Note that f is injective even on A^* because $f(A)$ is a suffix code. Thus f is circular by Lemma 6. Also, condition (iii) in Definition 1 holds because f is primitive. Thus $f \in \mathcal{N}$. □

Lemma 7 implies that, in particular, the Fibonacci morphism φ and the Thue–Morse morphism μ are in $\mathcal{N}$.

In Example 1 we saw that, in general, $\mathcal{S}$ is not necessarily closed under the shift map T for a general morphism f. The next lemma shows, however, that if $f \in \mathcal{N}$, this problem does not arise.

LEMMA 8. *If $f \in \mathcal{N}$, then $T(\mathcal{S}) \subseteq \mathcal{S}$. Thus $(\mathcal{S}, T)$ is a subshift.*

Proof. Let $\mathbf{s} = s_0 s_1 \cdots \in \mathcal{S}$; then $\pi(\mathbf{s}) \in \mathcal{X}$. Equation (2) says that $\pi(\mathbf{s}) = c(s_0) f(\pi(T\mathbf{s}))$, and so $f(\pi(T\mathbf{s})) \in \mathcal{X}$. Suppose that $\pi(T\mathbf{s}) \notin \mathcal{X}$.

The morphism f is circular because it is in $\mathcal{N}$. Let $\ell \geq 1$ be a synchronization delay for f. Now, observe first that $\pi(T\mathbf{s}) = \prod_{n \geq 1} f^{n-1}(c(s_n))$ and that $|f^{n-1}(c(s_n))| \to \infty$ as $n \to \infty$ because $f \in \mathcal{N}$. Observe then that $|f^{n-1}(c(s_n))|$ occurs in $f^{\omega}(\alpha)$ for every $n \geq 1$. Therefore there exists a word $zy \subset \pi(T\mathbf{s})$ such that $z \notin \mathcal{L}(\mathcal{X})$, $y \in \mathcal{L}(\mathcal{X})$, and $|y| \geq \ell$.

Next, consider the word $f(zy) \subset f(\pi(T\mathbf{s}))$. Since $f(y) \in \mathcal{L}(\mathcal{X})$ and $|f(y)| \geq \ell$, the word $f(y)$ has a synchronizing point (w_1, w_2). It follows that, since $y \in \mathcal{L}(\mathcal{X})$, there exist y_1, y_2 for which $y = y_1 y_2$, $f(y_1) = w_1$, and $f(y_2) = w_2$. On the other hand, $f(zy) \in \mathcal{L}(\mathcal{X})$ implies that we can write $f^{\omega}(\alpha) = ut\mathbf{x}$ such that $f(zy) \subset f(ut)$ and $f(y) \subset f(t)$. Thus the circularity of f implies that there exist t_1, t_2 such that $t = t_1 t_2$, the word w_1 is a suffix of $f(t_1)$, and w_2 is a prefix of $f(t_2)$. Thus $f(y_1)$ is a suffix of $f(t_1)$. Since $f(A)$ is a suffix code and f is injective, it follows that y_1 is a suffix of t_1, and furthermore that zy_1 is a suffix of ut_1. But then $z \in \mathcal{L}(\mathcal{X})$, contradicting the choice of z. Therefore $\pi(T\mathbf{s}) \in \mathcal{X}$ and so $T\mathbf{s} \in \mathcal{S}$. □

LEMMA 9. *If $f \in \mathcal{N}$, then the mapping $\pi : \mathcal{S} \to \mathcal{X}$ is injective.*

Proof. For every $u, v \in A^*$ and $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we have that $uf(\mathbf{x}) = vf(\mathbf{y})$ implies $u = v$ and $\mathbf{x} = \mathbf{y}$. This follows from the circularity and suffix code property of f (see also the proof of Lemma 8). Therefore if $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$ and $\pi(\mathbf{s}) = \pi(\mathbf{s}')$, then Lemma 3 gives

$$c(s_0) f(\pi(T\mathbf{s})) = c(s_0') f(\pi(T\mathbf{s}')),$$

so that $c(s_0) = c(s_0')$ and $\pi(T\mathbf{s}) = \pi(T\mathbf{s}')$. Thus $s_0 = s_0'$, and since $T\mathbf{s}, T\mathbf{s}' \in \mathcal{S}$ by Lemma 8, we can repeat the argument, obtaining $s_1 = s_1'$, $s_2 = s_2'$, ... Therefore $\mathbf{s} = \mathbf{s}'$. □

Remark 1. In Lemma 9 above, the assumption that f is circular is crucial. If $f : a^* \to a^*$ is defined by $f(a) = aa$, then $\mathcal{S} = S^{\mathbb{N}}$ while $\mathcal{X} = \{a^\omega\}$, so π is anything but injective! Nevertheless, f satisfies all conditions of $\mathcal{N}$, except circularity.

We are now ready to define the desired dynamics on $\mathcal{S}$.

THEOREM 1. *Suppose that $f \in \mathcal{N}$. Let $H : \mathcal{S} \to \mathcal{S}$ be the mapping given by $H = T \circ G$. Then $\pi \circ H = T \circ \pi$ and so $\pi : (\mathcal{S}, H) \to (\mathcal{X}, T)$ is a conjugacy.*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{H} & \mathcal{S} \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{X} & \xrightarrow{T} & \mathcal{X} \end{array}$$

Proof. Observe first that $H(\mathcal{S}) \subset \mathcal{S}$ by Lemma 8, so the definition of H is sound. The mapping π is surjective by Lemma 1 and injective by Lemma 9, so it is a bijection. Furthermore, π is continuous by Lemma 2. Finally, let us verify that $\pi \circ H = T \circ \pi$. Let $\mathbf{s} = s_0 s_1 \cdots \in \mathcal{S}$ with $s_i \in S$. If $|c(s_0)| \geq 2$, then we leave it to the reader to check that, by writing $c(s_0) = au$ with $a \in A$, we have

$$\pi \circ H(\mathbf{s}) = \pi \circ T \circ G(\mathbf{s}) = u f(c(s_1)) f^2(c(s_2)) \cdots = T \circ \pi(\mathbf{s}).$$

If $|c(s_0)| = 1$, then it is readily seen that $T \circ G(\mathbf{s}) = G \circ T(\mathbf{s})$. Using this, equation (4), and Lemma 3 in this order gives

$$\pi \circ H(\mathbf{s}) = \pi \circ T \circ G(\mathbf{s}) = \pi \circ G \circ T(\mathbf{s}) = f \circ \pi \circ T(\mathbf{s}) = T^{|c_0|} \circ \pi(\mathbf{s}) = T \circ \pi(\mathbf{s}),$$

and the proof is complete. □

The rest of this section is devoted to developing a few results for understanding the language of $\mathcal{S}$. They will be needed in the next sections that deal with the suffix conjugates of the Fibonacci and Thue–Morse subshifts.

If u is a finite non-empty word, we denote by $u^\flat$ and ${}^\natural u$ the words obtained from u by deleting its last and first letter, respectively.

If a finite word u is not in $\mathcal{L}(\mathcal{X})$, then u is called a *forbidden word* of $\mathcal{X}$. If both ${}^\natural u$ and $u^\flat$ are in $\mathcal{L}(\mathcal{X})$, then u is a *minimal forbidden word* of $\mathcal{X}$. There is a connection between the minimal forbidden words and the so-called bispecial factors of an infinite word. See a precise formulation of this in [**13**] and examples in §§3 and 4.

We say that a word $u \in S^*$ is a *cover* of a word $v \in A^*$ if $v \subset \pi(u)$. Furthermore, we say that the cover u is *minimal* if $v \not\subset \pi(u^\flat)$ and $v \not\subset f(\pi({}^\natural u))$. The latter expression comes from the identity $\pi(u) = c(u_0) f(\pi({}^\natural u))$, where u_0 is the first letter of u, given by Lemma 3.

Let $\mathcal{C}$ be the set of minimal covers of the minimal forbidden factors of $\mathcal{X}$.

LEMMA 10. *Suppose that $f \in \mathcal{N}$. Let $\mathbf{s} \in S^{\mathbb{N}}$. Then $\mathbf{s} \notin \mathcal{S}$ if and only if $\mathbf{s}$ has a factor in $\mathcal{C}$.*

Proof. Suppose that $\mathbf{s}$ has a factor in $\mathcal{C}$, so that $\mathbf{s} = pt\mathbf{s}'$ with $t \in \mathcal{C}$. If $\mathbf{s} \in \mathcal{S}$, then $T^{|p|}\mathbf{s} = t\mathbf{s}' \in \mathcal{S}$ by Lemma 8. But $\pi(t\mathbf{s}')$ has prefix $\pi(t)$, in which a forbidden word occurs by the definition of $\mathcal{C}$, a contradiction.

Conversely, suppose that $\mathbf{s} \notin \mathcal{S}$. Then $\pi(\mathbf{s}) \notin \mathcal{X}$, so there exists a minimal forbidden word v_0 of $\mathcal{X}$ occurring in $\pi(\mathbf{s})$. Let u_0 be the shortest prefix of $\mathbf{s}$ such that $v_0 \subset \pi(u_0)$. Then either u_0 is a minimal cover of v_0 or $v_0 \subset f(\pi({}^{\triangleleft}u_0))$. In the former case we are done, so suppose that the latter case holds. Then $v_0 \subset f(\pi(T\mathbf{s}))$ and so $\pi(T\mathbf{s})$ has a factor v_1 such that $v_0 \subset f(v_1)$ and $|v_1| \leq |v_0|$. Since $f(\mathcal{L}) \subset \mathcal{L}$, it follows that v_1 is a forbidden word of $\mathcal{X}$; by taking a factor of v_1 if necessary, we may assume that v_1 is also minimal. Let u_1 be the shortest prefix of $T\mathbf{s}$ such that $v_1 \subset \pi(u_1)$. Then either u_1 is a minimal cover of v_1 or $v_1 \subset f(\pi({}^{\triangleleft}u_1))$. In the former case $u_1 \in \mathcal{C}$ and so $\mathbf{s}$ has a factor u_1 in $\mathcal{C}$. In the latter case $v_1 \subset f(\pi(T^2\mathbf{s}))$, and we continue the process. This generates a sequence $v_0, v_1, \ldots$ of minimal forbidden words of $\mathcal{X}$ such that $v_n \subset f(\pi(T^{n+1}\mathbf{s}))$, $v_{n+1} \subset f(v_n)$, and $|v_{n+1}| \leq |v_n|$. Each letter $a \in A$ is growing because $f \in \mathcal{N}$, and therefore the words v_n are pairwise distinct. Thus the length restriction on the v_n implies that the sequence $v_0, v_1, \ldots$ is finite with a last element, say, v_k. The fact that there is no element v_{k+1} means that $T^{k+1}\mathbf{s}$ has a prefix u_k that is a minimal cover of v_k. Since $u_k \in \mathcal{C}$ then occurs also in $\mathbf{s}$, we are done. □

THEOREM 2. *Suppose that $f \in \mathcal{N}$ and that the set $\mathcal{C}$ of minimal covers of minimal forbidden words is a regular language. Then the language of $\mathcal{S}$ is regular. In particular, $(\mathcal{S}, T)$ is a sofic subshift.*

Proof. Since $\mathcal{C}$ is regular, so is the complement $S^* \setminus S^*\mathcal{C}S^*$, which we denote by L_0. Let M_0 be the minimal deterministic finite automaton accepting L_0. Modify M_0 by removing the states from which there are no arbitrarily long directed walks to accepting states. Remove also the corresponding edges and denote by M the non-deterministic finite automaton obtained. We claim that the language $\mathcal{L}(\mathcal{S})$ of $\mathcal{S}$ is the language $L(M)$ recognized by M.

If $w \in \mathcal{L}(\mathcal{S})$, then w is in $S^* \setminus S^*\mathcal{C}S^*$ by Lemma 10, so that it is accepted by M_0. Furthermore, since w has arbitrarily long extensions to the right that are also in $\mathcal{L}(\mathcal{S})$, each accepted by M_0 of course, it follows that w is accepted by M. Conversely, by the construction of M, if $w \in L(M)$, then there exists an infinite walk on the graph of M whose label contains w. The label of this infinite path is in $\mathcal{S}$. □

3. *The suffix conjugate of the Fibonacci subshift*

Recall the Fibonacci morphism φ for which $\mathtt{0} \mapsto \mathtt{01}$ and $\mathtt{1} \mapsto \mathtt{0}$, the Fibonacci word $\mathbf{f} = \varphi^{\omega}(\mathtt{0})$, and the Fibonacci subshift $(\mathcal{X}_{\varphi}, T)$. The suffix conjugate $(\mathcal{S}_{\varphi}, H_{\varphi})$ of the Fibonacci subshift is guaranteed to exist by Lemma 7 and Theorem 1. The goal of this section is to give a characterization for $\mathcal{S}_{\varphi}$ and H_{φ}, and it will be achieved in Theorem 3.

The set of suffixes of φ is $S' = \{\mathtt{0}, \mathtt{1}, \mathtt{01}\}$, and we define a bijection c between $S = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$ and S' by $c(\mathtt{0}) = \mathtt{0}$, $c(\mathtt{1}) = \mathtt{1}$, and $c(\mathtt{2}) = \mathtt{01}$. In this case we have $\mathcal{S}_{\varphi} \subset \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}^{\mathbb{N}}$.

We will now continue by finding a characterization for the set $\mathcal{C}_\varphi$ of minimal covers of minimal forbidden words of the Write $f_n = \varphi^{n-1}(\mathtt{0})$ for all $n \geq 1$, so that in particular $f_1 = \mathtt{0}$ and $f_2 = \mathtt{01}$. For $n \geq 2$, we let p_n be the word defined by the relation $f_n = p_n ab$, where $ab \in \{\mathtt{01}, \mathtt{10}\}$. Then $p_2 = \varepsilon$ and $p_3 = \mathtt{0}$. The words p_n are known as the *bispecial factors* of the Fibonacci word, and they possess the following well-known and easily established properties.

- For all $n \geq 2$, we have
$$f_n f_{n-1} = p_{n+1}ab \quad \text{and} \quad f_{n-1} f_n = p_{n+1}ba, \tag{8}$$
where $ab = \mathtt{10}$ for even n and $ab = \mathtt{01}$ for odd n.
- For all $n \geq 2$, we have $\varphi(p_n)\mathtt{0} = p_{n+1}$.

The minimal forbidden words of the Fibonacci word $\mathbf{f}$ can be expressed in terms of the bispecial factors p_n as follows [**13**]. For every $n \geq 2$, write

$$d_n = \begin{cases} \mathtt{1}p_n\mathtt{1} & \text{for } n \text{ even},\\ \mathtt{0}p_n\mathtt{0} & \text{for } n \text{ odd}. \end{cases}$$

Then a word is a minimal forbidden word of $\mathbf{f}$ if and only if it equals d_n for some $n \geq 2$. The first few d_n are $\mathtt{11}$, $\mathtt{000}$, and $\mathtt{10101}$.

If x is a finite word and y a finite or infinite word, we write $(x <_p y)$ $x \leq_p y$ to indicate that x is a (proper) prefix of y. We say two finite words x, y are *prefix compatible* if one of $x \leq_p y$ or $y \leq_p x$ holds.

We will skip the easy proof of the following lemma.

LEMMA 11. *Let $x, y \in \{\mathtt{0}, \mathtt{1}\}^+$ and $k \geq 1$. Then $\varphi^k(x) <_p \varphi^k(y)$ implies $x^\flat <_p y^\flat$.*

LEMMA 12. *Let $x, y \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}^*$ and suppose that $\pi(x) <_p \pi(y)$. Then either $x^\flat <_p y^\flat$ or $x = u\mathtt{01}$ and $y = u\mathtt{2}s$ for some $u \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}^*$ and non-empty $s \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}^+$.*

Proof. Suppose that $x^\flat$ is not a prefix of $y^\flat$. Then $x = uat$ and $y = ubs$ with distinct letters $a, b \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$ and non-empty words $t, s \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}^+$. Lemma 3 applied to finite words gives
$$\pi(x) = \pi(u)\varphi^{|u|}(\pi(at)) \quad \text{and} \quad e(y) = \pi(u)\varphi^{|u|}(\pi(bs)).$$
Thus $\pi(x) <_p \pi(y)$ implies $\varphi^{|u|}(\pi(at)) <_p \varphi^{|u|}(\pi(bs))$, so that by Lemma 11, we have $\pi(at)^\flat <_p \pi(bs)^\flat$, or
$$c(a)\varphi(\pi(t))^\flat <_p c(b)\varphi(\pi(s))^\flat.$$
Since $a \neq b$, it follows that $a = \mathtt{0}$, $b = \mathtt{2}$, and $\varphi(\pi(t))^\flat = \varepsilon$. The last identity implies $t = \mathtt{1}$; therefore $x = u\mathtt{01}$ and $y = u\mathtt{2}s$. □

The next lemma has a straightforward verification, which we will omit.

LEMMA 13. *We have $d_3 = \pi(\mathtt{01})\mathtt{0}$ and $d_4 = \pi(\mathtt{10})\mathtt{01}$. For all $n \geq 0$, we have*
$$\pi(\mathtt{021}^{2n}\mathtt{2}) = d_{2n+5}\mathtt{1} \quad \text{and} \quad \pi(\mathtt{121}^{2n+1}\mathtt{2}) = d_{2n+6}\mathtt{0}.$$

LEMMA 14. *The forbidden word $d_2 = \mathtt{11}$ does not have covers. The minimal covers of d_3 are the words in $\mathtt{01}(\mathtt{0} + \mathtt{1} + \mathtt{2})$. The minimal covers of d_4 are the words in $(\mathtt{1} + \mathtt{2})\mathtt{0}(\mathtt{0} + \mathtt{1} + \mathtt{2})$. For other forbidden words, we have the following. Let $n \geq 0$.*

(i) *The minimal covers of* d_{2n+5} *are*

$$\mathtt{0}\mathtt{2}\mathtt{1}^{2n}(\mathtt{2}+\mathtt{00}+\mathtt{01}+\mathtt{02}). \tag{9}$$

(ii) *The minimal covers of* d_{2n+6} *are*

$$(\mathtt{1}+\mathtt{2})\mathtt{2}\mathtt{1}^{2n+1}(\mathtt{2}+\mathtt{00}+\mathtt{01}+\mathtt{02}).$$

Proof. We leave verifying the claims on d_2, d_3, and d_4 to the reader. The displayed words are minimal covers because they are obtained from the clearly minimal words in Lemma 13 by modifying the first and the last two letters in obvious ways.

To prove that this collection is exhaustive, suppose that u is a minimal cover of d_{2n+5}. Then $d_{2n+5} \subset \pi(u)$, and since d_{2n+5} is not a factor of $\varphi(\pi({}^{\triangleleft}u))$, it follows that u can be written as $u = ax$ with $a \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$ such that $d_{2n+5} \leq_p \pi(bx)$ for some $b \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. Noticing that d_{2n+5} starts with with $\mathtt{00}$, we actually must have $a = b = \mathtt{0}$, and so $d_{2n+5} \leq_p \pi(u)$. Lemma 13 says that then $\pi(\mathtt{0}\mathtt{2}\mathtt{1}^{2n}\mathtt{2}) <_p \pi(u)$, so that $\mathtt{0}\mathtt{2}\mathtt{1}^{2n} <_p u^{\flat}$ by Lemma 12. It is readily verified using (8) that $\pi(\mathtt{0}\mathtt{2}\mathtt{1}^{2n}\mathtt{1})$ is not prefix compatible with d_{2n+5}, and thus either $\mathtt{0}\mathtt{2}\mathtt{1}^{2n}\mathtt{0} \leq_p u$ or $\mathtt{0}\mathtt{2}\mathtt{1}^{2n}\mathtt{2} \leq_p u$. This observation and the minimality of u show that the words in (9) are exactly all the minimal covers of d_{2n+5}.

The case for d_{2n+6} can be handled in the same way, the only difference being that since d_{2n+6} starts with $\mathtt{10}$, the letters a and b may differ, but then $\{a, b\} = \{\mathtt{1}, \mathtt{2}\}$. □

THEOREM 3. *The language* $\mathcal{L}(\mathcal{S}_\varphi)$ *of the suffix conjugate* $(\mathcal{S}_\varphi, H_\varphi)$ *of the Fibonacci subshift* $(\mathcal{X}_\varphi, T)$ *is regular. An infinite word* $\mathbf{s} \in S^{\mathbb{N}}$ *is in* $\mathcal{S}_\varphi$ *if and only if it is the label of an infinite walk on the graph depicted in Figure 1. The mapping* $H_\varphi \colon \mathcal{S}_\varphi \to \mathcal{S}_\varphi$ *is given by*

$$H_\varphi(\mathbf{s}) = \begin{cases} \mathtt{1}\mathbf{z} & \text{if } \mathbf{s} = \mathtt{2}\mathbf{z}, \\ \lambda(x\mathtt{2})\mathbf{z} & \text{if } \mathbf{s} = ax\mathtt{2}\mathbf{z} \text{ with } a \in \{\mathtt{0}, \mathtt{1}\},\ x \in \{\mathtt{0}, \mathtt{1}\}^*, \\ \lambda(\mathbf{z}) & \text{if } \mathbf{s} = a\mathbf{z} \text{ with } a \in \{\mathtt{0}, \mathtt{1}\} \text{ and } \mathbf{z} \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}}, \end{cases}$$

where λ *is the morphism given by* $\lambda(\mathtt{1}) = \mathtt{0}$, $\lambda(\mathtt{0}) = \mathtt{2}$, *and* $\lambda(\mathtt{2}) = \mathtt{21}$.

Proof. Lemma 14 says that the set $\mathcal{C}_\varphi$ of all minimal covers of minimal forbidden words of $\mathbf{f}$ is regular. Thus Theorem 2 tells us that $\mathcal{L}(\mathcal{S}_\varphi)$ is regular. Following the proof of that theorem, we first construct the minimal deterministic automaton† accepting the language $S^* \setminus S^*\mathcal{C}_\varphi S^*$ and then remove the states and edges that cannot be on the path of an infinite walk through accepting states. The result is given in Figure 1(a). Notice that the label of each walk starting from state q_0 can be obtained from a walk starting from states q_1, q_3, or q_4. The state q_2 is superfluous for the same reason. The removal of states q_0 and q_2 and the corresponding edges yields in the graph in Figure 1.

Using equation (6) for constructing the mapping G and then recalling the definition $H = T \circ G$, the given formula for H_φ is readily verified. This completes the proof. □

Our last goal for this section is to prove Theorem 4. To that end, let us first state a well-known property of the Fibonacci subshift.

† This automaton and the one in the proof Theorem 5 were computed using Petri Salmela's FAFLA Python package for finite automata and formal languages. See `http://coyote.dy.fi/~pesasa/fafla/`

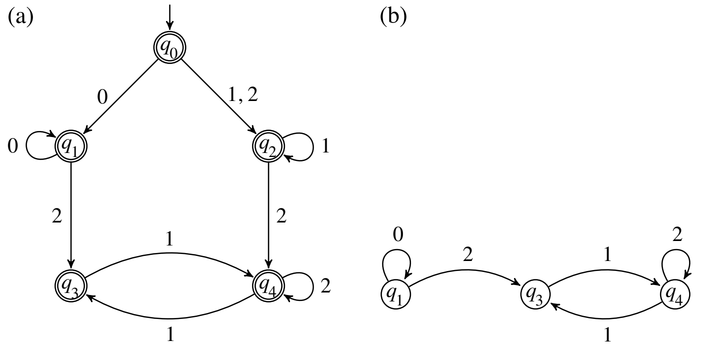

FIGURE 1. The suffix conjugate of the Fibonacci subshift. (a) An NFA accepting the language of $\mathcal{L}(\mathcal{S}_\varphi)$. (b) A graph for the sequences in $\mathcal{S}_\varphi$.

LEMMA 15. *If* $\mathbf{z}$ *has two* T*-preimages in* $\mathcal{X}_\varphi$*, then* $\mathbf{z} = \mathbf{f}$*.*

Proof. If $\mathbf{z}$ has two T-preimages, then $\mathtt{0}\mathbf{z}$, $\mathtt{1}\mathbf{z} \in \mathcal{X}_\varphi$. This means that all prefixes of $\mathbf{z}$ are so-called left special factors of the Fibonacci word $\mathbf{f}$. The unique word in $\mathcal{X}_\varphi$ with this property is $\mathbf{f}$; see, for example, [**11**, Ch. 2]. □

We say that an infinite word $\mathbf{x}$ is in the *strictly positive orbit* of $\mathbf{z}$ if $T^k\mathbf{z} = \mathbf{x}$ for some $k > 0$, and that $\mathbf{x}$ is the *strictly negative orbit* of $\mathbf{z}$ if $T^k\mathbf{x} = \mathbf{z}$ for some $k > 0$. An infinite word $\mathbf{x}$ is said to have a *tail* $\mathbf{z}$ if $\mathbf{x} = u\mathbf{z}$ for some finite word u.

Since $\mathbf{f} = \mathtt{01}\varphi(\mathtt{1})\varphi^2(\mathtt{1}) \cdots$, we have $\mathbf{f} = \pi(\mathtt{21}^\omega)$. Thus Theorem 3 gives

$$T\mathbf{f} = T \circ \pi(\mathtt{21}^\omega) = \pi \circ H_\varphi(\mathtt{21}^\omega) = \pi(\mathtt{1}^\omega).$$

Similarly,

$$T^2\mathbf{f} = \pi(\mathtt{0}^\omega) \quad \text{and} \quad T^3\mathbf{f} = \pi(\mathtt{2}^\omega)$$

The word $T^2\mathbf{f}$ divides the shift orbit of the Fibonacci word in the following sense.

THEOREM 4. *Let* $\mathbf{s} \in \mathcal{S}_\varphi$*. Then:*
(i) $\pi(\mathbf{s})$ *is in the strictly positive orbit of* $T^2\mathbf{f}$ *if and only if* $\mathbf{s}$ *has a tail* $\mathtt{2}^\omega$*;*
(ii) $\pi(\mathbf{s})$ *is in the strictly negative orbit of* $T^2\mathbf{f}$ *if and only if* $\mathbf{s}$ *has a tail* $\mathtt{1}^\omega$*.*

Proof. We begin by proving (i). If $\pi(\mathbf{s})$ is in the strictly positive orbit of $T^2\mathbf{f}$, then there exists $k > 0$ such that

$$\pi(\mathbf{s}) = T^{k+2}\mathbf{f} = T^k \circ \pi(\mathtt{0}^\omega) = \pi \circ H_\varphi^k(\mathtt{0}^\omega) = \pi \circ H_\varphi^{k-1}(\mathtt{2}^\omega).$$

Since π is injective, $\mathbf{s} = H_\varphi^{k-1}(\mathtt{2}^\omega)$. Thus we see from the characterization of H_φ given in Theorem 3 that $\mathbf{s}$ has a tail $\mathtt{2}^\omega$.

Conversely, suppose that $\mathbf{s}$ has a tail $\mathtt{2}^\omega$. Then there exists an integer $m \geq 0$ such that both $\pi(\mathbf{s})$ and $T^3\mathbf{f}$ have a tail $f^m(\pi(\mathtt{2}^\omega))$. This implies that there exist finite words u, v and $\mathbf{z} \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}}$ such that $\pi(\mathbf{s}) = u\mathbf{z}$ and $T^3\mathbf{f} = v\mathbf{z}$ and the only common suffix of u and v is the empty word ε.

Case 1: $u \neq \varepsilon$ and $v \neq \varepsilon$. Then $0\mathbf{z}$, $1\mathbf{z} \in \mathcal{X}_\varphi$, and so by Lemma 15, we have $\mathbf{z} = \mathbf{f}$. But $\mathbf{f}$ cannot be a tail of $T^3\mathbf{f}$ because $\mathbf{f}$ is aperiodic, a contradiction.

Case 2: $u \neq \varepsilon$ and $v = \varepsilon$. Then both $T^{|u|-1} \circ \pi(\mathbf{s})$ and $T^2\mathbf{f}$ are T-preimages of $T^3\mathbf{f}$, so that $T^{|u|-1} \circ \pi(\mathbf{s}) = T^2\mathbf{f}$ again by Lemma 15. Thus

$$\pi(0^\omega) = T^2\mathbf{f} = T^{|u|-1} \circ \pi(\mathbf{s}) = \pi \circ H_\varphi^{|u|-1}(\mathbf{s}),$$

from which we get $H_\varphi^{|u|-1}(\mathbf{s}) = 0^\omega$ by the injectivity of π. But this is not possible because $\mathbf{s}$ has a tail 2^ω and Theorem 3 says that H_φ preserves such tails.

Case 3: $u = \varepsilon$. Then $\pi(\mathbf{s})$ is in the strictly positive orbit of $T^2\mathbf{f}$, and this is what we wanted to prove.

Let us now prove (ii). If $\pi(\mathbf{s})$ is in the negative orbit of $T^2\mathbf{f}$, then $T^k\pi(\mathbf{s}) = T^2\mathbf{f}$ for some $k > 0$. Then

$$\pi(1^\omega) = T\mathbf{f} = T^{k-1} \circ \pi(\mathbf{s}) = \pi \circ H_\varphi^{k-1}(\mathbf{s}),$$

so that $H_\varphi^{k-1}(\mathbf{s}) = 1^\omega$ by the injectivity of π. The characterization of H_φ given in Theorem 3 shows, then, that $\mathbf{s}$ must have a tail 1^ω.

Conversely, suppose that $\mathbf{s}$ has a tail 1^ω. Since $T\mathbf{f} = \pi(1^\omega)$, there exists an integer $m \geq 0$ such that both $\pi(\mathbf{s})$ and $T\mathbf{f}$ have a common tail $f^m(\pi(1^\omega))$. Thus $\pi(\mathbf{s}) = u\mathbf{z}$ and $T\mathbf{f} = v\mathbf{z}$ for some finite words u, v and $\mathbf{z} \in \{0, 1\}^{\mathbb{N}}$ such that the only common suffix of u and v is the empty word ε.

Case 1: $v \neq \varepsilon$, $u \neq \varepsilon$. Then $0\mathbf{z}$, $1\mathbf{z} \in \mathcal{X}_\varphi$, so that $\mathbf{z} = \mathbf{f}$, contradicting the fact that $\mathbf{f}$ cannot be a tail of $T\mathbf{f}$.

Case 2: $v \neq \varepsilon$, $u = \varepsilon$. Then

$$\pi(\mathbf{s}) = T^{|v|} \circ T\mathbf{f} = T^{|v|} \circ \pi(1^\omega) = \pi \circ H_\varphi^{|v|}(1^\omega) = \pi \circ H_\varphi^{|v|-1}(0^\omega),$$

so that $\mathbf{s} = H_\varphi^{|v|-1}(0^\omega)$. Once again, the characterization of H_φ shows that 1^ω cannot be a tail of $\mathbf{s}$, a contradiction.

Case 3: $v = \varepsilon$. Then $\pi(\mathbf{s})$ is in the strictly negative orbit of $T^2\mathbf{f}$, which is want we wanted to prove. □

4. *The suffix conjugate of the Thue–Morse subshift*

Let μ be the Thue–Morse morphism $0 \mapsto 01$, $1 \mapsto 10$ and $\mathbf{t} = \mu^\omega(0)$ the Thue–Morse word. Let $\mathcal{X}_\mu$ denote the shift orbit closure of $\mathbf{t}$, so that the Thue–Morse subshift is $(\mathcal{X}_\mu, T)$. In this section we will characterize its suffix conjugate $(\mathcal{S}_\mu, H_\mu)$, which is guaranteed to exist by Lemma 7 and Theorem 1.

Here the set of suffixes is $S' = \{0, 1, 01, 10\}$, and $S = \{0, 1, 2, 3\}$. We set c to be the bijection between S and S' given by

$$c(0) = 0, \quad c(1) = 1, \quad c(2) = 01, \quad c(3) = 10.$$

The minimal forbidden words of the Thue–Morse word are 000, 111,

$$0\mu^{2n}(010)0, \quad 0\mu^{2n}(101)0, \quad 1\mu^{2n}(010)1, \quad 1\mu^{2n}(101)1$$

and

$$\texttt{1}\mu^{2n+1}(\texttt{010})\texttt{0}, \quad \texttt{1}\mu^{2n+1}(\texttt{101})\texttt{0}, \quad \texttt{0}\mu^{2n+1}(\texttt{010})\texttt{1}, \quad \texttt{0}\mu^{2n+1}(\texttt{101})\texttt{1}$$

for all $n \geq 0$; see [**13**, **16**].

Let us introduce a shorthand. For $x, y, z \in \{\texttt{0}, \texttt{1}\}$ and $k \geq 0$, we write

$$\gamma(k, x, y, z) = x\mu^k(y\overline{y}y)z.$$

Here the overline notation $\bar{\cdot}$ swaps $\texttt{0}$s and $\texttt{1}$s. The minimal forbidden words of $\mathbf{t}$ can then be written as xxx and

$$\gamma(2n, x, x, x), \quad \gamma(2n, x, \overline{x}, x), \quad \gamma(2n+1, x, x, \overline{x}), \quad \gamma(2n+1, x, \overline{x}, \overline{x}), \tag{10}$$

for all $n \geq 0$ and $x \in \{\texttt{0}, \texttt{1}\}$. Furthermore,

- $\mu(\gamma(k, x, y, z)) = x\gamma(k+1, \overline{x}, y, z)\overline{z}$, and
- $\gamma(k, x, y, z)$ is a forbidden word if and only if $\gamma(k-1, \overline{x}, y, z)$ is a forbidden word, where $k \geq 1$.

The mapping λ defined in equations (3) and (5) becomes

$$\lambda(\texttt{0}) = \texttt{2}, \quad \lambda(\texttt{1}) = \texttt{3}, \quad \lambda(\texttt{2}) = \texttt{21}, \quad \lambda(\texttt{3}) = \texttt{30}.$$

The next lemma is analogous to Lemma 12 for the Fibonacci subshift. The proof is straightforward, so we will skip it.

LEMMA 16. *Let $x, y \in \{\texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}\}^*$ and suppose that $\pi(x) <_p \pi(y)$. Then one of the following holds:*

(i) $x^\flat <_p y^\flat$;

(ii) *$x = uz\overline{z}$ and $y = u\lambda(z)s$, where $z \in \{\texttt{0}, \texttt{1}\}$ and s has prefix z or $\lambda(z)$;*

(iii) *$x = u\lambda(z)z$ and $y = uzs$, where $z \in \{\texttt{0}, \texttt{1}\}$ and s has prefix $\overline{z}\overline{z}$ or $\overline{z}\lambda(\overline{z})$.*

Our next goal is to characterize the minimal covers of the minimal forbidden words of $\mathcal{X}_\mu$. We begin with the next lemma, whose easy verification is left to the reader.

LEMMA 17. *Let $x \in \{\texttt{0}, \texttt{1}\}$. The forbidden words xxx and $\gamma(0, x, x, x)$ do not have covers. For other forbidden words, we have the following.*

(i) *The minimal covers of $\gamma(0, x, \overline{x}, x)$ are in*

$$(x + \lambda(\overline{x}))\overline{x}(\overline{x} + \lambda(\overline{x})) \quad \textit{and} \quad \lambda(x)x(x + \lambda(x)).$$

(ii) *The minimal covers of $\gamma(1, x, x, \overline{x})$ are in*

$$(x + \lambda(\overline{x}))x\overline{x}(\overline{x} + \lambda(\overline{x})) \quad \textit{and} \quad (x + \lambda(\overline{x}))\lambda(x)(x + \lambda(x)).$$

(iii) *The minimal covers of $\gamma(1, x, \overline{x}, \overline{x})$ are in*

$$(x + \lambda(\overline{x}))\overline{x}x(\overline{x} + \lambda(\overline{x})) \quad \textit{and} \quad (x + \lambda(\overline{x}))\overline{x}\lambda(x).$$

We will characterize the minimal covers of the remaining forbidden words in Lemma 19 with the help of the following result, whose straightforward proof we omit.

LEMMA 18. *For $x, y, z \in \{\mathtt{0}, \mathtt{1}\}$ and $k \geq 2$, we have*

$$\gamma(k, x, y, z) = \pi(x\lambda(y)\overline{y}^{k-2}\lambda(\overline{y}))z. \tag{11}$$

Furthermore, $x\lambda(y)\overline{y}^{k-1}w$ is a minimal cover of $\gamma(k, x, y, \overline{y})$ with

$$\gamma(k, x, y, \overline{y}) <_p \pi(x\lambda(y)\overline{y}^{k-1}w) \tag{12}$$

if and only if $w \in \{y, \lambda(y)\}$.

LEMMA 19. *Let $x, y \in \{\mathtt{0}, \mathtt{1}\}$ and $k \geq 2$. A word is a minimal cover of $\gamma(k, x, y, y)$ if and only if it is in*

$$(x + \lambda(\overline{x}))\lambda(y)\overline{y}^{k-2}\lambda(\overline{y})(y + \lambda(y)). \tag{13}$$

A word is a minimal cover of $\gamma(k, x, y, \overline{y})$ if and only if it is in

$$(x + \lambda(\overline{x}))\lambda(y)\overline{y}^{k-2}\left[\lambda(\overline{y})(\overline{y} + \lambda(\overline{y})) + \overline{y}(y + \lambda(y))\right]. \tag{14}$$

Proof. We see from (11) that the words in (13) really are minimal covers of $\gamma(k, x, y, y)$. (Notice here that $c(x)$ is a suffix of $c(\lambda(\overline{x}))$ and $c(y)$ is a prefix of $c(\lambda(y))$.) Further, we see from (11) and (12) that the words in (14) really are minimal covers of $\gamma(k, x, y, \overline{y})$.

Let us show the converse. Let $u \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}\}^*$ be a minimal cover of $\gamma(k, x, y, z)$ with $z \in \{y, \overline{y}\}$. Then there exist $a, b \in \{x, \lambda(\overline{x})\}$ and $t \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}\}^*$ such that $u = at$ and $\gamma(k, x, y, z) \leq_p \pi(bt)$. Since both possibilities for a are accounted for in (13) and (14), we may assume that $a = b$, and so $u = bt$. Then (11) gives

$$\pi(x\lambda(y)\overline{y}^{k-2}\lambda(\overline{y})) = \gamma(k, x, y, z)z^{-1} <_p \pi(u). \tag{15}$$

Now Lemma 16 applies, and since its options (ii) and (iii) are clearly not possible here, we get

$$x\lambda(y)\overline{y}^{k-2} <_p u^\flat,$$

and so we have $u = x\lambda(y)\overline{y}^{k-2}w$ for some $w \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}\}^+$. The minimality of u implies $|w| = 2$. Equation (15) and Lemma 3 then imply

$$\mu^k(\pi(\lambda(\overline{y})))z <_p \mu^k(\pi(w))$$

or, equivalently,

$$\pi(\lambda(\overline{y}))z <_p \pi(w).$$

If $z = y$, then $w \in \lambda(\overline{y})(y + \lambda(y))$. If $z = \overline{y}$, then either

$$w \in \overline{y}(y + \lambda(y)) \quad \text{or} \quad w \in \lambda(\overline{y})(\overline{y} + \lambda(\overline{y})),$$

and this completes the proof. □

THEOREM 5. *The language $\mathcal{L}(\mathcal{S}_\mu)$ of the suffix conjugate $(\mathcal{S}_\mu, H_\mu)$ of the Thue–Morse subshift $(\mathcal{X}_\mu, T)$ is regular. An infinite word $\mathbf{s} \in S^{\mathbb{N}}$ is in $\mathcal{S}_\mu$ if and only if it is the label of an infinite walk on the graph depicted in Figure 2. The mapping $H_\mu \colon \mathcal{S}_\mu \to \mathcal{S}_\mu$ is given by*

$$H_\mu(\mathbf{s}) = \begin{cases} \mathtt{1}\mathbf{z} & \text{if } \mathbf{s} = \mathtt{2}\mathbf{z}, \\ \mathtt{0}\mathbf{z} & \text{if } \mathbf{s} = \mathtt{3}\mathbf{z}, \\ \lambda(\mathbf{z}) & \text{if } \mathbf{s} = a\mathbf{z} \text{ with } a \in \{\mathtt{0}, \mathtt{1}\} \text{ and } \mathbf{z} \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}}, \\ \lambda(x\mathtt{2})\mathbf{z} & \text{if } \mathbf{s} = ax\mathtt{2}\mathbf{z} \text{ with } a \in \{\mathtt{0}, \mathtt{1}\}, x \in \{\mathtt{0}, \mathtt{1}\}^*, \\ \lambda(x\mathtt{3})\mathbf{z} & \text{if } \mathbf{s} = ax\mathtt{3}\mathbf{z} \text{ with } a \in \{\mathtt{0}, \mathtt{1}\}, x \in \{\mathtt{0}, \mathtt{1}\}^*, \end{cases}$$

where λ is the morphism given by $\lambda(\mathtt{0}) = \mathtt{2}$, $\lambda(\mathtt{1}) = \mathtt{3}$, $\lambda(\mathtt{2}) = \mathtt{21}$, and $\lambda(\mathtt{3}) = \mathtt{30}$.

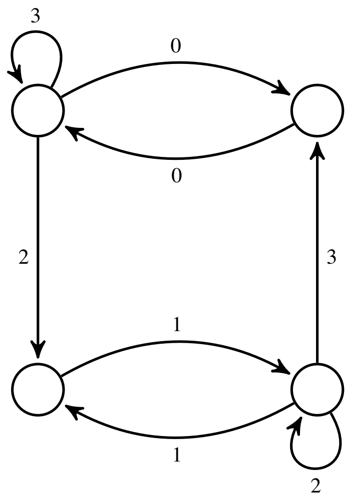

FIGURE 2. The suffix conjugate of the Thue–Morse subshift.

Proof. The set $\mathcal{C}_\mu$ is obtained from equation (10) using Lemmas 17 and 19. From these it is clear that $\mathcal{C}_\mu$ is a regular language. Thus $\mathcal{L}(\mathcal{S}_\mu)$ is regular by Theorem 2. Taking the steps outlined in the proof of that theorem and removing the superfluous states and edges, as in the proof of Theorem 3, we get the graph depicted in Figure 2. Finally, the values of H_μ are obtained directly from the definition $H = T \circ G$ and equation (6). □

Using the characterization of H_μ from Theorem 5, it is readily verified that

$$\mathbf{t} = \mu^\omega(\texttt{0}) = \pi(\texttt{21}^\omega), \quad T\mathbf{t} = \pi(\texttt{1}^\omega), \quad T^2\mathbf{t} = \pi(\texttt{3}^\omega).$$

Our last goal for this section is to establish Theorem 6, for which we need the next lemma.

LEMMA 20. *If* $\mathbf{x}$ *has two* T*-preimages in* $\mathcal{X}_\mu$*, then either* $\mathbf{x} = \mathbf{t}$ *or* $\mathbf{x} = \bar{\mathbf{t}}$.

Proof. Since $\mathbf{x}$ is aperiodic, it must have infinitely many prefixes p such that both $p\texttt{0}$ and $p\texttt{1}$ occur in $\mathbf{x}$. Since $\texttt{0}p$ and $\texttt{1}p$ occur in $\mathbf{x}$ as well, the words p are so-called bispecial factors of $\mathbf{t}$, whose form is known [**2**, Proposition 4.10.5]. They are

$$\varepsilon, \quad \texttt{0}, \quad \texttt{1}, \quad \mu^m(\texttt{01}), \quad \mu^m(\texttt{10}), \quad \mu^m(\texttt{010}), \quad \mu^m(\texttt{101})$$

for $m \geq 0$. Therefore either $\mathbf{x} = \mu^\omega(\texttt{0}) = \mathbf{t}$ or $\mathbf{x} = \mu^\omega(\texttt{1}) = \bar{\mathbf{t}}$. □

We say that an infinite word $\mathbf{x}$ is in the *positive orbit* of $\mathbf{z}$ if $T^k\mathbf{z} = \mathbf{x}$ for some $k \geq 0$, and that $\mathbf{x}$ is the *negative orbit* of $\mathbf{z}$ if $T^k\mathbf{x} = \mathbf{z}$ for some $k \leq 0$. Notice that in the previous section we used the notions strictly positive and strictly negative orbits. The next result and its proof are analogous to Theorem 4, once we have established Lemma 20. The details are omitted.

THEOREM 6. *Let* $\mathbf{s} \in \mathcal{S}_\mu$*. Then:*

(i) $\pi(\mathbf{s})$ *is in the positive orbit of* $T^2\mathbf{t}$ *if and only if* $\texttt{3}^\omega$ *is a tail of* $\mathbf{s}$*;*

(ii) $\pi(\mathbf{s})$ *is in the negative orbit of* $T\mathbf{t}$ *if and only if* $\texttt{1}^\omega$ *is a tail of* $\mathbf{s}$.

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