Cyclically t-complementary uniform hypergraphs

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Abstract

A *cyclically t-complementary k-hypergraph* is a k-uniform hypergraph with vertex set V and edge set E for which there exists a permutation $\theta \in Sym(V)$ such that the sets $E, E^{\theta}, E^{\theta^2}, \ldots, E^{\theta^{t-1}}$ partition the set of all k-subsets of V. Such a permutation θ is called a (t,k)-*complementing permutation*. The cyclically t-complementary k-hypergraphs are a natural and useful generalization of the self-complementary graphs, which have been studied extensively in the past due to their important connection to the graph isomorphism problem.

For a prime p, we characterize the cycle type of the (p^r, k)-complementing permutations $\theta \in Sym(V)$ which have order a power of p. This yields a test to determine whether a permutation in $Sym(V)$ is a (p^r, k)-complementing permutation, and an algorithm for generating all of the cyclically p^r-complementing k-hypergraphs of order n, for feasible n, up to isomorphism. We also obtain some necessary and sufficient conditions on the order of these structures. This generalizes previous results due to Ringel, Sachs, Adamus, Orchel, Szymański, Wojda, Zwonek, and Bernaldez.

Key words: Self-complementary hypergraph, Uniform hypergraph, t-complementing permutation, Cyclically t-complementary hypergraph.
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1 Introduction

For a finite set V and a positive integer k, let $V^{(k)}$ denote the set of all k-subsets of V. A *hypergraph* with vertex set V and edge set E is a pair (V, E), in which V is a finite set and E is a collection of subsets of V. A hypergraph (V, E) is called k-*uniform* (or a k-*hypergraph*) if E is a subset of $V^{(k)}$. The parameters k

and $|V|$ are called the *rank* and the *order* of the k-hypergraph, respectively. The vertex set and the edge set of a hypergraph X will often be denoted by $V(X)$ and $E(X)$, respectively. Note that a 2-hypergraph is a *graph*. An *isomorphism* between k-hypergraphs X and X' is a bijection $\phi : V(X) \to V(X')$ which induces a bijection from $E(X)$ to $E(X')$. If such an isomorphism exists, the hypergraphs X and X' are said to be *isomorphic*.

A k-hypergraph $X = (V, E)$ is *cyclically t-complementary* if there exists a permutation θ on V such that the sets $E, E^{\theta}, E^{\theta^2}, \ldots, E^{\theta^{t-1}}$ partition $V^{(k)}$. We denote the set E^{θ^i} by E_i. Note that $E_i^{\theta} = E_{i+1}$ for $i = 0, 1, \ldots, t-2$ and $E_{t-1}^{\theta} = E_0 = E$. Such a permutation θ is called a *(t,k)-complementing permutation*, and it gives rise to a family of t isomorphic k-hypergraphs $\{X_i = (V, E_i) : i = 0, 1, \ldots, t-1\}$ which partition the complete k-hypergraph on V, and which are permuted cyclically under the action of θ.

The cyclically t-complementatry k-hypergraphs have been previously defined and studied in the cases where $t = 2$ or $k = 2$, and there is some overlap and some contradiction between the terminology used in these cases. The cyclically 2-complementary 2-hypergraphs are the self-complementary graphs. In 1978, M.J. Colbourn and C.J. Colbourn [3] showed that one of the most important problems in graph theory, the graph isomorphism problem, is polynomially equivalent to the problem of determining whether two self-complementary graphs are isomorphic. Since then, there has been a great deal of research into self-complementary graphs. A good reference on self-complementary graphs and their generalizations was written by A. Farrugia [4]. The cyclically 2-complementary k-hypergraphs are the self-complementary k-hypergraphs studied in [5, 8, 10, 11, 12], and in the terminology of these papers the $(2, k)$-complementing permutations are their corresponding 'k-complementing permutations', or 'antimorphisms'. The cyclically t-complementary graphs (2-hypergraphs) are the t-complementary graphs, or t-c graphs, studied in [1, 2] and in the terminology of these papers the $(t, 2)$-complementing permutations are their corresponding 't-complementing permutations' or 't-c permutations'.

Whether or not a permutation θ is (t, k)-complementing depends entirely on the cycle type of θ. The cycle type of the $(2, 2)$-complementing permutations were characterized in [6, 7] and the cycle types of the $(2, 3)$- and $(2, 4)$-complementing permutations were characterized in [8] and [9], respectively. Quite recently, these earlier results were generalized to characterize the cycle type of the $(2, k)$-complementing permutations in [5, 10, 12], and the cycle type of the $(t, 2)$-complementing permutations was determined in [1, 2]. In Theorem 3.2, we generalize both of these new results and characterize the cycle type of the (q, k)-complementing permutations which have order a power of p, where $q = p^r$ is a prime power. We will show that this is sufficient to characterize all of the (q, k)-complementing permutations for these q, and we obtain necessary and sufficient conditions on the order of a q-complementary k-hypergraph.

In Section 2, we will prove some useful facts about (t, k)-complementing permutations, and then in Section 3, we will use these facts to prove the main result in Theorem 3.2. This yields Corollary 3.3, which gives a method for testing any permutation algorithmically to determine whether it is (q,k)-complementing, and a method for generating all of the cyclically q-complementary k-hypergraphs of order n, for feasible n. In Section 4, we obtain Corollary 4.1, which gives necessary and sufficient conditions on the order of a q-complementary k-hypergraph in the case where q is a prime power, and these conditions simplify in the case where q is prime.

2 The (t,k)-complementing permutations

We have the following natural characterization of the (t,k)-complementing permutations.

Lemma 2.1. *Let V be a finite set, let k and t be positive integers, and let $\theta \in Sym(V)$. Then the following three statements are equivalent:*

(1) θ is a (t,k)-complementing permutation.

(2) $A^{\theta^j} \neq A$ for $j \not\equiv 0 \pmod{t}$, for all $A \in V^{(k)}$.

(3) The sequence $A, A^{\theta}, A^{\theta^2}, A^{\theta^3}, \ldots$ has length divisible by t, for all $A \in V^{(k)}$.

Proof: Suppose θ is a (t,k)-complementing permutation. Then there is a k-hypergraph $X = (V,E)$ such that $E_0, E_1, \ldots, E_{t-1}$ partitions $V^{(k)}$, where $E_i = E^{\theta^i}$. Let $A \in V^{(k)}$. Then $A \in E_i$ for exactly one $i \in \{0,1,\ldots,t-1\}$. If $j \not\equiv 0 \pmod{t}$, then $A^{\theta^j} \in E_i^{\theta^j} = E_{(i+j) \bmod t} \neq E_i$. Hence $A^{\theta^j} \notin E_i$, and so in particular $A^{\theta^j} \neq A$. Hence (1) implies (2).

Suppose (2) holds. Let j be the length of a sequence in (3). Then $A^{\theta^j} = A$, and so (2) implies that $j \equiv 0 \pmod{t}$. Hence (2) implies (3).

Suppose (3) holds. To show that (3) implies (1), we describe a simple algorithm which takes a permutation θ satisfying (3) as input, and returns the nonempty set $\mathcal{H}_\theta$ of all cyclically t-complementary k-hypergraphs X on V that have θ as a (t,k)-complementing permutation. This algorithm was previously described in the case where $k=2$ by Adamus et al [1].

Algorithm 2.2. Let $\theta \in Sym(V)$ satisfy (3).

(I) Construct the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_m$ of θ on $V^{(k)}$. Each orbit $\mathcal{O}_j$ has the form
$$A, A^{\theta}, A^{\theta^2}, A^{\theta^3}, \ldots$$
where $A \in V^{(k)}$, and hence each orbit $\mathcal{O}_j$ is a sequence in (3).

(II) For each $j \in \{1,2,\ldots,m\}$, choose $i \in \{0,1,\ldots,t-1\}$ and let E_i^j denote the set of k-sets of the form $A^{\theta^{tz+i}}$ in the orbit $\mathcal{O}_j$ constructed in (I), where z is an integer. Since (3) holds, each orbit $\mathcal{O}_j$ has length divisible by t. Thus, within each orbit $\mathcal{O}_j$, θ maps E_i^j to E_{i+1}^j for each $i = 0,1,\ldots,t-2$, and θ maps E_{t-1}^j to E_0^j.

(III) Let E be a subset of $V^{(k)}$ that contains exactly one of the sets E_0^j, E_1^j, $E_2^j, \ldots, E_{t-1}^j$ constructed in (II) for each $j \in \{1, 2, \ldots, m\}$. Then $X = (V, E)$ is a cyclically t-complementary k-hypergraph. Moreover, if there are m orbits of θ on $V^{(k)}$, then there are t^m different choices for the edge set E, and the t^m different choices for E generate the set $\mathcal{H}_\theta$ of all t^m cyclically t-complementary k-hypergraphs on V for which θ is a (t, k)-complementing permutation.

∎

In the next lemma, we obtain some useful properties of (t, k)-complementing permutations. For a permutation θ on a set V, the symbol $|\theta|$ denotes the order of θ in $Sym(V)$.

Lemma 2.3. *Let V be a finite set, and let s, t and k be positive integers such that* $\gcd(t, s) = 1$.

(1) A permutation $\theta \in Sym(V)$ is a (t, k)-complementing permutation if and only if θ^s is a (t, k)-complementing permutation.

(2) The order of a (t, k)-complementing permutation is divisible by t.

(3) If $q = p^r$ is a prime power, every cyclically q-complementary k-hypergraph has a (q, k)-complementing permutation with order a power of p.

Proof:

(1) If $\theta \in Sym(V)$ is a (t, k)-complementing permutation, then there is a cyclically t-complementary k-hypergraph $X = (V, E)$ such that the sets $E_0, E_1, \ldots, E_{t-1}$ partition $V^{(k)}$, where $E_i = E^{\theta^i}$. Consider the sequence

$$E_0, E_s, E_{2s}, E_{3s}, \ldots, E_{(t-1)s},$$

where each subscript is taken modulo t. If $is \equiv js \pmod{t}$ for some i, j where $0 \le i < j \le t-1$, then since $\gcd(s, t) = 1$ we must have $i \equiv j \pmod{t}$, a contradiction. Hence the subscripts $0, s, 2s, 3s, \ldots, (t-1)s$ are pairwise incongruent modulo t, and hence the sets $E_0, E_s, E_{2s}, E_{3s}, \ldots, E_{(t-1)s}$ (with subscripts taken modulo t) also partition $V^{(k)}$. That is, the sets

$$E, E^{\theta^s}, E^{(\theta^s)^2}, \ldots, E^{(\theta^s)^{t-1}}$$

partition $V^{(k)}$, and so θ^s is a also (t, k)-complementing permutation of X.

Conversely, suppose that θ^s is a (t, k)-complementing permutation. Then Proposition 2.1 guarantees that each orbit of θ^s on $V^{(k)}$ has cardinality congruent to 0 modulo t . Observe that each orbit of θ^s on $V^{(k)}$ is contained in an orbit of θ on $V^{(k)}$. Also, every k-subset in an orbit of θ on $V^{(k)}$ must certainly lie in an orbit of θ^s on $V^{(k)}$. Since the orbits of θ^s on $V^{(k)}$ are pairwise disjoint, it follows that every orbit of θ on $V^{(k)}$ is a union of pairwise disjoint orbits of θ^s on $V^{(k)}$, each of which has cardinality divisible by t. Hence every orbit of θ on $V^{(k)}$ has cardinality divisible by t, and so by Proposition 2.1, θ is a (t,k)-complementing permutation.

(2) This follows directly from Lemma 2.1(2).

(3) Let $X = (V, E)$ be a cyclically q-complementary k-hypergraph. Then X has a (q,k)-complementing permutation $\sigma \in Sym(V)$, and by part (2), the order of σ is divisible by q, and hence by p. Thus $|\sigma| = p^a b$ for a positive integer a and an integer b such that p does not divide b. Since $\gcd(b,q) = 1$, part (1) implies that $\theta = \sigma^b$ is also a (q,k)-complementing permutation of X, and its order is $|\theta| = p^a$. ∎

3 Cycle types of (q,k)-complementing permutations

For a prime power $q = p^r$, Theorem 3.2 gives a characterization of the cycle types of the (q,k)-complementing permutations which have order equal to a power of p, in terms of the base-p representation of k. We will make use of the following technical lemma to prove Theorem 3.2.

Lemma 3.1. [5] *Let ℓ and p be positive integers, where $p \geq 2$. Let $a_0, a_1, \ldots, a_{\ell-1}$ be nonnegative integers such that $\sum_{i=0}^{\ell-1} a_i p^i \geq p^\ell$. Then there exists a sequence of integers $c_0, c_1, \ldots, c_{\ell-1}$, where $0 \leq c_i \leq a_i$, such that $\sum_{i=0}^{\ell-1} c_i p^i = p^\ell$.* ∎

To state and prove Theorem 3.2, we require some terminology and notation. We will denote the *base-p representation* of an integer k by $b(p,k)$, where $b(p,k)$ is the vector $(b_m, b_{m-1}, \ldots, b_1, b_0)_p$. This is, $b(p,k)$ is the vector such that $k = \sum_{i=0}^m b_i p^i$, $b_m \neq 0$, and $b_i \in \{0, 1, \ldots, p-1\}$ for $0 \leq i \leq m$. The *support* of the base-p representation $b = b(p,k)$ is the set $\{i \in \{0,1,2,\ldots,m\} : b_i \neq 0\}$, and is denoted by $supp(b)$. For positive integers m and n, let $n_{[m]}$ denote the unique integer in $\{0, 1, \ldots, m-1\}$ such that $n \equiv n_{[m]} (\mathrm{mod}\ m)$. For a permutation θ on a set V, an *invariant set* of θ is a subset of V which is fixed set-wise by θ.

Theorem 3.2. *Let V be a finite set and let k be a positive integer such that $k \leq |V|$. Let $q = p^r$ be a prime power, and let $b = b(p,k) = (b_m, b_{m-1}, \ldots, b_2, b_1, b_0)_p$ be the base-p representation of k. Let $\theta \in Sym(V)$ be a permutation whose order is a power of p. For an integer $m \geq 0$, let A_m denote those points of V contained in cycles of θ of length at most p^m. Then θ is a (q,k)-complementing permutation if and only if there is $\ell \in supp(b)$ such that*

$$|A_{\ell+r-1}| < k_{[p^{\ell+1}]}.$$

Proof: $(\Rightarrow)$
Claim 1: If $\theta \in Sym(V)$ has order a power of p and $|A_\ell| \geq k_{[p^{\ell+1}]}$ for all $\ell \in supp(b)$, then θ has an invariant set of size k.

Proof of Claim 1: Suppose that $\theta \in Sym(V)$ has order a power of p, and that $|A_\ell| \geq k_{[p^{\ell+1}]}$ for all $\ell \in supp(b)$. Every cycle of θ has length a power of p. Let a_i denote the number of cycles of θ of length p^i. If $a_i \geq b_i$ for every $i \in supp(b)$, then there would be an invariant set of θ of cardinality $\sum_{i \in supp(b)} b_i p^i = k$, as claimed. Hence we may assume that, for some $i \in supp(b)$, $a_i < b_i$. Let

$$L = \{i \in supp(b) : a_i < b_i\}. \tag{1}$$

Then $L \neq \emptyset$. Since $L \subseteq supp(b)$, we have $|A_\ell| \geq k_{[p^{\ell+1}]}$ for all $\ell \in L$.

Now $|A_\ell| = \sum_{i=0}^{\ell} a_i p^i$. Note that $k_{[p^{\ell+1}]} = \sum_{i=0}^{\ell} b_i p^i$. Thus, by assumption, $\sum_{i=0}^{\ell} a_i p^i \geq \sum_{i=0}^{\ell} b_i p^i$ for all $\ell \in L$. Let

$$L = \{\ell_1, \ell_2, \ldots, \ell_z\}$$

where $\ell_1 < \ell_2 < \cdots < \ell_z$.

- **Claim 1A:** Let $x \in \{1, 2, \ldots, z\}$. If $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i p^i$ for all $j \in \{1, 2, \ldots, x\}$, then $\theta|_{A_{\ell_x}}$ has an invariant set of size $\sum_{i=0}^{\ell_x} b_i p^i$.

  Proof of Claim 1A: The proof is by induction on x.

  Base Step: If $x = 1$ and $|A_{\ell_1}| \geq \sum_{i=0}^{\ell_1} b_i p^i$, then

  $$|A_{\ell_1}| = \sum_{i=0}^{\ell_1} a_i p^i \geq \sum_{i=0}^{\ell_1} b_i p^i. \tag{2}$$

  Since ℓ_1 is the smallest element of the set L defined in (1), it follows that $a_i \geq b_i$ for $0 \leq i \leq \ell_1 - 1$ and $a_{\ell_1} < b_{\ell_1}$. Thus (2) implies that

  $$\sum_{i=0}^{\ell_1 - 1} (a_i - b_i) p^i \geq (b_{\ell_1} - a_{\ell_1}) p^{\ell_1}$$

  holds with $a_i - b_i \geq 0$ for all $i = 1, 2, \ldots, \ell_1 - 1$. Applying Lemma 3.1 $b_{\ell_1} - a_{\ell_1}$ times, we obtain a sequence $c_0, c_1, \ldots, c_{\ell_1 - 1}$ such that $0 \leq c_i \leq (a_i - b_i)$ for $0 \leq i \leq \ell_1 - 1$, and

  $$\sum_{i=0}^{\ell_1 - 1} c_i p^i = (b_{\ell_1} - a_{\ell_1}) p^{\ell_1}.$$

  Now let $\hat{a}_i = b_i + c_i$ for $1 \leq i \leq \ell_1 - 1$ and let $\hat{a}_{\ell_1} = a_{\ell_1}$. Then

  $$0 \leq \hat{a}_i = b_i + c_i \leq b_i + (a_i - b_i) = a_i$$

for $1 \le i \le \ell_1 - 1$ and hence $0 \le \hat{a}_i \le a_i$ for $0 \le i \le \ell_1$. Moreover

$$\sum_{i=0}^{\ell_1} \hat{a}_i p^i = \sum_{i=0}^{\ell_1-1} b_p^i + \sum_{i=0}^{\ell_1-1} c_i p^i + a_{\ell_1} p^{\ell_1} = \sum_{i=0}^{\ell_1-1} b_i p^i + (b_{\ell_1} - a_{\ell_1}) p^{\ell_1} + a_{\ell_1} p^{\ell_1}$$

and hence

$$\sum_{i=0}^{\ell_1} \hat{a}_i p^i = \sum_{i=0}^{\ell_1} b_i p^i.$$

The sum $\sum_{i=0}^{\ell_1} \hat{a}_i p^i$ is the sum of the lengths of a collection of cycles of $\theta|_{A_{\ell_1}}$, and hence it is the size of an invariant set of $\theta|_{A_{\ell_1}}$. Thus $\theta|_{A_{\ell_1}}$ has an invariant set of size $\sum_{i=0}^{\ell_1} b_i p^i$, as required.

Induction Step: Let $2 \le x \le z$ and assume that if $|A_{\ell_j}| \ge \sum_{i=0}^{\ell_j} b_i p^i$ for all $j \in \{1, 2, \ldots, x-1\}$, then $\theta|_{A_{\ell_{x-1}}}$ has an invariant set of size $\sum_{i=0}^{\ell_{x-1}} b_i p^i$. Now suppose that $|A_{\ell_j}| \ge \sum_{i=0}^{\ell_j} b_i p^i$ for all $j \in \{1, 2, \ldots, x\}$. Then certainly $|A_{\ell_j}| \ge \sum_{i=0}^{\ell_j} b_i p^i$ for all $j \in \{1, 2, \ldots, x-1\}$, and so by the induction hypothesis, $\theta|_{A_{\ell_{x-1}}}$ has an invariant set of size $\sum_{i=0}^{\ell_{x-1}} b_i p^i$. By the definition of L in (1), $a_i \ge b_i$ for $\ell_{x-1} < i < \ell_x$. Thus $\theta\, |_{A_{\ell_x - 1}}$ has an invariant set of size $\sum_{i=0}^{\ell_x - 1} b_i p^i$. This implies that there is a sequence of integers $c_0, c_1, \ldots, c_{\ell_x - 1}$ such that $0 \le c_i \le a_i$ for $0 \le i \le \ell_x - 1$, and

$$\sum_{i=0}^{\ell_x - 1} c_i p^i = \sum_{i=0}^{\ell_x - 1} b_i p^i. \tag{3}$$

Since $|A_{\ell_x}| \ge \sum_{i=0}^{\ell_x} b_i p^j$, we have

$$\sum_{i=0}^{\ell_x} a_i p^i \ge \sum_{i=0}^{\ell_x} b_i p^i. \tag{4}$$

Since $\ell_x \in L$, $a_{\ell_x} < b_{\ell_x}$, so (3) and (4) together imply that

$$\sum_{i=0}^{\ell_x - 1} (a_i - c_i) p^i \ge (b_{\ell_x} - a_{\ell_x}) p^i.$$

Since $a_i - c_i \ge 0$ for $0 \le i \le \ell_x - 1$, we can apply Lemma 3.1 $b_{\ell_x} - a_{\ell_x}$ times to obtain a sequence of non-negative integers $d_0, d_1, \ldots, d_{\ell_x - 1}$ such that $0 \le d_i \le (a_i - c_i)$ for $0 \le i \le \ell_x - 1$, and

$$\sum_{i=0}^{\ell_x - 1} d_i p^i = (b_{\ell_x} - a_{\ell_x}) p^{\ell_x}.$$

Now let $\hat{a}_i = c_i + d_i$ for $0 \le i \le \ell_x - 1$ and let $\hat{a}_{\ell_x} = a_{\ell_x}$. Then one can check that $0 \le \hat{a}_i \le a_i$ for $0 \le i \le \ell_x$, and

$$\sum_{i=0}^{\ell_x} \hat{a}_i p^i = \sum_{i=0}^{\ell_x} b_i p^i.$$

Since $\sum_{i=0}^{\ell_x} \hat{a}_i p^i$ is the sum of the lengths of a collection of cycles of $\theta|_{A_{\ell_x}}$, we conclude that $\theta|_{A_{\ell_x}}$ has an invariant set of size $\sum_{i=0}^{\ell_x} b_i p^i$, as required.

Hence by the principle of mathematical induction, Claim 1A holds for all $x \in \{1, 2, \ldots, z\}$. **(Claim 1A)**∎

Now applying Claim 1A with $x = z$, we observe that since $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i p^i$ for all $j \in \{1, 2, \ldots, z\}$, $\theta|_{A_{\ell_z}}$ has an invariant set of size $\sum_{i=0}^{\ell_z} b_i p^i$. But since ℓ_z is the largest element of L, θ contains b_ℓ cycles of length p^ℓ for all $\ell \in supp(b)$ with $\ell_z < \ell \leq m$, and hence θ contains an invariant set of size $\sum_{i=0}^{m} b_i p^i = k$. This proves Claim 1. **(Claim 1)** ∎

Now suppose that $\theta \in Sym(V)$ is a (q, k)-complementing permutation with order a power of p. For an integer j, let A_ℓ^j denote the set of elements of V which lie in cycles of θ^j of length at most p^ℓ. Note that $A_\ell^j = A_{\ell+a}$ where a is the largest integer such that p^a divides j.

If $|A_{\ell+r-1}| \geq k_{[p^{\ell+1}]}$ for all $\ell \in supp(b)$, then for $j = p^{r-1}$ we have $|A_\ell^j| = |A_{\ell+r-1}| \geq k_{[p^{\ell+1}]}$ for all $\ell \in supp(b)$. Hence Claim 1 implies that θ^j has an invariant set of size k. But since $q = p^r$, $j = p^{r-1} \not\equiv 0 \pmod{q}$, and so the fact that θ^j fixes a k-subset of V contradicts Lemma 2.1. We conclude that $|A_{\ell+r-1}| < k_{[p^{\ell+1}]}$ for some $\ell \in supp(b)$, as claimed.

($\Leftarrow$) Let $\theta \in Sym(V)$ with order a power of p and suppose that there is $\ell \in supp(b)$ such that $|A_{\ell+r-1}| < k_{[p^{\ell+1}]}$. Let j be an integer such that $j \not\equiv 0 \pmod{q}$. Then $j = p^a b$ for integers a and b where $0 \leq a < r$ and p does not divide b. Thus $|A_\ell^j| = |A_{\ell+a}| \leq |A_{\ell+r-1}| < k_{[p^{\ell+1}]}$. This implies that θ^j does not have an invariant set of size k. Since j was chosen arbitrarily, we conclude that $A^{\theta^j} \neq A$ for all $j \not\equiv 0 \pmod{q}$ and all $A \in V^{(k)}$, and so Proposition 2.1 implies that θ is a (q, k)-complementing permutation. ∎

Lemma 2.3 and Theorem 3.2 together yield the following characterization of (q, k)-complementing permutations.

Corollary 3.3. *Let k be a positive integer, let $q = p^r$ be a prime power, let $b = b(p, k)$ be the base-p representation of k, and let V be a finite set. A permutation $\sigma \in Sym(V)$ is a (q, k)-complementing permutation if and only if $|\sigma| = jp^i$ for some integers i and j such that $i \geq 1$ and $\gcd(p, j) = 1$, and $\theta = \sigma^j$ satisfies the condition of Theorem 3.2 for some $\ell \in supp(b)$.* ∎

Corollary 3.3 and the conditions of Theorem 3.2 can be used to test a permutation algorithmically to determine if it is a (q, k)-complementing permutation.

If $q = p^r$ is a prime power, Lemma 2.3(3) guarantees that every cyclically q-complementary k-hypergraph has an (q, k)-complementing permutation which has order a power of p. Hence we can generate all of the cyclically q-complementary k-hypergraphs of order n, up to isomorphism, by applying Algorithm 2.2 to find $\mathcal{H}_\theta$ for every permutation θ in $Sym(n)$ satisfying the conditions of Theorem 3.2. Moreover, if we just wish to generate at least one representative of each *isomorphism class* of cyclically q-complementary k-hypergraphs of order n, it suffices to apply Algorithm 2.2 to one permutation θ from each conjugacy class of permutations in $Sym(n)$ satisfying the conditions of Theorem 3.2.

4 Necessary and sufficient conditions on order

In this section, we present necessary and sufficient conditions on the order n of a cyclically q-complementary k-hypergraph when $q = p^r$ is a prime power. Since Lemma 2.3(3) guarantees that every cyclically q-complementary k-hypergraph has a (q, k)-complementing permutation with order equal to a power of p, Theorem 3.2 immediately implies the following necessary and sufficient conditions on the order of these structures.

Corollary 4.1. *Let k and n be positive integers, $k \leq n$, let $q = p^r$ be a prime power, and let b be the base-p representation of k. There exists a cyclically q-complementary k-hypergraph of order n if and only if there is $\ell \in supp(b)$ such that*

$$n_{[p^{\ell+r}]} < k_{[p^{\ell+1}]}. \tag{5}$$

∎

Corollary 4.2. *Let k and n be positive integers, $k \leq n$, let p be a prime, and let b be the base-p representation of k. There exists a cyclically p-complementary k-hypergraph of order n if and only if*

$$n_{[p^{\ell+1}]} < k_{[p^{\ell+1}]} \quad \text{for some } \ell \in supp(b).$$

Proof: Set $q = p^1$ in Corollary 4.1. Then $r = 1$ and so the only choice for $a \in \{0, 1, \ldots, r-1\}$ is $a = 0$. Thus there exists a cyclically p-complementary k-hypergraph if and only if condition (5) holds with $a = 0$ for some $\ell \in supp(b)$. ∎

When the rank k is within $p - 1$ of a multiple of a power of a prime p, then Corollary 4.2 yields the following more transparent necessary and sufficient conditions on the order of a cyclically p-complementary k-hypergraph.

Corollary 4.3. *Let ℓ be a positive integer and let p be prime.*

1. *If $k = b_\ell p^\ell$ for $0 < b_\ell < p$, then there exists a cyclically p-complementary k-hypergraph of order n if and only if $n_{[p^{\ell+1}]} < k$.*

2. *If $k = b_\ell p^\ell + b_0$ where $0 < b_0, b_\ell < p$, then there exists a cyclically p-complementary k-hypergraph of order n if and only if $n_{[p]} < b_0$ or $n_{[p^{\ell+1}]} < k$.*

Proof:

1. In this case $supp(b) = \{\ell\}$, and so Corollary 4.2 implies that there exists a cyclically p-complementary k-hypergraph of order n if and only if

$$n_{[p^{\ell+1}]} < k_{[p^{\ell+1}]}. \tag{6}$$

Since $k = b_\ell p^\ell < p^{\ell+1}$, $k_{[p^{\ell+1}]} = k$, and so (6) is equivalent to $n_{[p^{\ell+1}]} < k$.

2. In this case $supp(b) = \{0, \ell\}$ and so Corollary 4.2 implies that there exists a cyclically p-complementary k-hypergraph of order n if and only if $n_{[p]} < k_{[p]}$ or $n_{[p^{\ell+1}]} < k_{[p^{\ell+1}]}$. Since $k = b_\ell p^\ell + b_0$, $k_{[p]} = b_0$ and $k_{[p^{\ell+1}]} = k$, the result follows. ∎

In the case where $k = \sum_{i=0}^{s} (p-1)p^{\ell+i}$ for a nonnegative integer s, the condition of Corollary 4.2 holds for the largest integer in the support of the base-p representation of k, as the next result shows.

Corollary 4.4. *Let r, s and ℓ be nonnegative integers, let p be prime, and suppose that $k = \sum_{i=0}^{s} (p-1)p^{\ell+i}$. Then there exists a cyclically p-complementary k-hypergraph of order n if and only if $n_{[p^{\ell+s+1}]} < k$.*

Proof: Suppose that there exists a cyclically p-complementary k-hypergraph of order n, and let b be the base-p representation of k. Then

$$supp(b) = \{\ell, \ell+1, \ldots, \ell+s\},$$

and so Corollary 4.2 guarantees that

$$n_{[p^{\ell+j+1}]} < k_{[p^{\ell+j+1}]}, \tag{7}$$

for some $j \in \{0, 1, 2, \ldots, s\}$. If (7) holds for some $j < s$, then the fact that

$$n_{[p^{\ell+(j+1)+1}]} \leq (p-1)p^{\ell+j+1} + n_{[p^{\ell+j+1}]}$$

implies that

$$n_{[p^{\ell+(j+1)+1}]} < (p-1)p^{\ell+j+1} + k_{[p^{\ell+j+1}]}. \tag{8}$$

Now since $(p-1)p^{\ell+j+1} + k_{[p^{\ell+j+1}]} = (p-1)p^{\ell+j+1} + \sum_{i=0}^{j} (p-1)p^{\ell+i} = k_{[p^{\ell+(j+1)+1}]}$, (8) implies that

$$n_{[p^{\ell+(j+1)+1}]} < k_{[p^{\ell+(j+1)+1}]},$$

and hence (7) also holds for $j+1$. Thus, by induction on j, the fact that (7) holds for some $j \in \{0, 1, \ldots, s\}$ implies that (7) holds for $j = s$. Hence $n_{[p^{\ell+s+1}]} < k_{[p^{\ell+s+1}]} = k$.

Conversely, Corollary 4.2 guarantees that there exists a cyclically p-complementary k-hypergraph of order n for every integer n such that $n_{[p^{\ell+s+1}]} < k_{[p^{\ell+s+1}]} = k$. ∎

Corollary 4.5. *If $k = p^\ell - 1$, then there exists a cyclically p-complementary k-hypergraph if and only if $n_{[p^\ell]} < k$.*

Proof: Since $k = \sum_{i=0}^{\ell-1}(p-1)p^i$, the result follows directly from Corollary 4.4. ∎

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